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ABSTRACT

We characterize symmetry transformations of Lagrangian extremals generating "on shell" conservation laws. We relate symmetry transformations of extremals to Jacobi fields and study symmetries of higher variations by proving that a pair given by a symmetry of the lth variation of a Lagrangian and by a Jacobi field of the sth variation of the same Lagrangian (with $s < l$) is associated with an "off shell" conserved current. The conserved current associated with two symmetry transformations is constructed, and as a case of study, its expression for invariant sets of Yang–Mills connections on Minkowski space-times is obtained.

I. INTRODUCTION

The description of fundamental interactions in physics as fields associated with the action of Lie groups on maps between manifolds has been the cornerstone of the last century. Indeed, within this picture, fields are (local) maps between manifolds adapted to a fibration (which distinguishes independent from dependent variables and their peculiarity when changing coordinates), i.e., they are sections of fibrations, having the additional structure of a bundle (fields take values in a manifold, which is the type fiber of the bundle). In particular, it is well known that due to their invariance properties, physical fields can be described as sections of bundles associated with principal bundles, and the configuration bundles are, then, the so-called gauge-natural bundle; see, e.g., Refs. 1 and 2.

It is noteworthy that the variational derivation of field equations is an intrinsic operation strictly related just to the fibration structure and its prolongation up to a given order.[1,3–5] This approach had several important developments, in particular when combined with invariance properties (the geometric formulation of the Noether theorems, specifically).

Furthermore important is now the possibility of a systematic formulation of higher variations (see *Proposition III.6* and *Theorem IV.11*), which can be interpreted as variations of suitable "deformed" Lagrangians.[6]

Combined with symmetry considerations, this approach extends to field theory the somehow analogous concept of the so-called Sarlet–Cantrijn higher order (dynamical) Noether symmetries in mechanics[7] (which we can think as a kind of higher order Noether–Bessel–Hagen symmetries of the Lagrangian[8,9,36]). The present paper refers directly to *canonical Noether symmetries*, i.e., symmetries of the Lagrangian, which we can describe, roughly speaking, as dynamical Noether symmetries up to divergences. Restricting to mechanics, we recognize second order Sarlet–Cantrijn dynamical Noether symmetries up to divergences as related to second order canonical Noether currents in our approach.

Higher variations are of interest in theoretical physics, in particular concerning variations of currents;[10,11] for applications of the second variation in gravitational theory in this context, see, e.g., Ref. 12.

Lagrangian symmetries and symmetries of Euler–Lagrange equations were called by Andrzej Trautman[5] invariant transformations and generalized invariant transformations, respectively, and they were characterized as particular kinds of what he called *symmetry transformations*, i.e., transformations of extremals into extremals of the same Euler–Lagrange equations.

Indeed, it is well known that a symmetry of Euler–Lagrange equations (generalized invariant transformations) is also a symmetry transformation of their solutions (extremals), i.e., a transformation preserving the property of a field (a section of the configuration bundle on space–time) being an extremal.[5,13]

The inverse, in general, is not true: symmetry transformations of solutions of equations could not be symmetries of the equations themselves. A related result stating that a Lagrangian "dragged" along symmetry transformations of its own extremals has the same extremals as the original Lagrangian (and an inverse statement stating that a transformation dragging a given Lagrangian in a Lagrangian having the same extremals is a symmetry transformation of the extremals) was obtained in Ref. 14 (see Theorem IV.3).

We focus on conservation laws by explicating the relation among higher variations of Lagrangians, symmetry transformations of extremals, Jacobi fields, and conserved currents. We characterize symmetry transformations of extremals as particular transformations of the Euler–Lagrange forms to source forms vanishing along extremals of the original Lagrangian and specifically as Jacobi fields *along extremals* (*Theorem IV.5*).

Compared with generalized symmetry transformations (i.e., transformations leaving invariant the Euler–Lagrange form of a Lagrangian), such transformations provide a weaker invariance property since the Euler–Lagrange form is not invariant under their action, although it is transformed to a source form having the same extremals. Therefore, the equations change, but the solutions of the one equation are also the solutions of the second *and vice versa*.

By explicating the relation among the variation of an Euler–Lagrange form with the second variation of a Lagrangian and with the Jacobi morphisms (*Proposition III.3* and *Remark IV.14*), we prove that with this sort of weaker invariance is anyway associated a conserved current and, in particular, that this current can be identified as a Noether current for a Lagrangian "deformed" by a symmetry transformation of extremals and associated with (or generated by) a symmetry transformation of extremals. More specifically, symmetry transformations of extremals generate conserved currents along the extremals themselves. Indeed, *Corollary IV.15* states the existence of a weak (i.e., along extremals) conservation law for any couple of (infinitesimal) generators of (vertical) symmetry transformations.

As an explicit example, we write the expressions of the on shell conserved current generated by couples of symmetry transformations of extremals.

II. CONTACT STRUCTURE, GEOMETRIC INTEGRATION BY PARTS, AND THE "REPRESENTATION" PROBLEM

Throughout this paper, we work with a fibration $\pi : Y \to X$, where Y is an $m + n$-dimensional manifold and X is an n-dimensional manifold. When choosing coordinates, we will always pick fibered coordinates (x^i, y^σ) defined over open subsets $\pi^{-1}(U) \subset Y$, where U is an open subset of the base. We also set ds to be the local expression of a volume element $dx^1 \wedge \cdots \wedge dx^n$ on X and $ds_i = \frac{\partial}{\partial x^i} \rfloor ds$. Given this geometric setting, physical fields are encoded as local sections of the surjective submersion $\pi : Y \to X$.

Let $\Omega_q(J^k Y)$ denote the module of q-forms on the set $J^k Y$ of equivalence classes of (local) sections of the fibration having a contact of order k in a point. Note that $J^k Y$ has the structure of a differentiable manifold and the structure of a fibration $\pi_k : J^k Y \to X$, the prolongation of order k of $\pi : Y \to X$.

A prolongation map j^k assigns to (local) sections of the fibration $\pi : Y \to X$ (local) sections of the fibration $\pi_k : J^k Y \to X$. If γ is a section of π, $j^k\gamma$ is defined as the map assigning to $x \in X$ the k-jet $j^k_x\gamma$ of σ at x. A differential q-form $\alpha \in \Omega_q(J^k Y)$ is called contact if $j^k\gamma^*(\alpha) = 0$ for all sections γ of π. It is easy to see that forms ω locally given as

$$\omega^\sigma_{j_1\ldots j_h} = dy^\sigma_{j_1\ldots j_h} - y^\sigma_{j_1\ldots j_h i}dx^i$$

for $0 \le h < k$ are indeed contact one-forms.

In particular, $(dx^i, \omega^\sigma, \omega^\sigma_{j_1}, \ldots, \omega^\sigma_{j_1\ldots j_{k-1}}, dy^\sigma_{j_1\ldots j_k})$ is a local basis for one-forms on $J^k Y$. Contact forms on a fixed prolongation space $J^k Y$ generate an ideal of the exterior algebra; this is usually called the contact structure induced by the affine bundle projections $\pi_{k,k-1} : J^k Y \to J^{k-1} Y$; see, e.g., Refs. 13 and 4. It is important to note that if α is contact, so is $d\alpha$; on the other hand, the ideal of the exterior algebra of forms on $J^k Y$ generated by the forms $\omega^\sigma_{j_1}, \ldots, \omega^\sigma_{j_1\ldots j_{k-1}}$ is not closed under exterior derivation.

For every form $\rho \in \Omega_q(J^k Y)$, by the contact structure, we obtain the canonical decomposition,[13]

$$\pi^*_{k+1,k}(\rho) = p_0\rho + p_1\rho + \cdots + p_q\rho,$$

where $p_0\rho$ is a form that is horizontal on X (and so is often denoted by $h\rho$), while $p_i\rho$ is an i-contact q-form, which is a form generated by wedge products containing exactly i-contact one-forms. We remark that if $q > n$, every q-form ρ is contact; then, we call it *strongly* contact if $p_{q-n} = 0$. The contact structure also induces the splitting of the exterior differential $\pi^*_{k+1,k}d\rho = d_H\rho + d_V\rho$ in the so-called horizontal and vertical differentials, given by $d_H\rho = \sum_{l=0}^{q} p_l dp_l\rho$ and $d_V\rho = \sum_{l=0}^{q} p_{l+1} dp_l\rho$, respectively.

According to Ref. 15, we define the *formal derivative* with respect to the ith coordinate, $i = 1, \ldots, n$, by an abuse of notation also denoted by d_i, as an operator acting on forms. Explicitly, we require d_i to be the usual total derivative on zero-forms, to commute with the exterior derivative, and to satisfy the Leibniz rule with respect to the wedge product. We see that $d_H\rho = (-1)^q d_i\rho \wedge dx^i$ if ρ is a q-form. On the basis one-forms associated with the contact structure, we have $d_i dx^j = 0$, $d_i\omega^\sigma_{j_1\ldots j_r} = \omega^\sigma_{j_1\ldots j_r i}$, and $d_i dy^\sigma = dy^\sigma_i$. Note that d_i induces a vector field on J^kY along $\pi_{k+1,k}$; we still denote it by d_i and refer to it as the formal derivative.

In the following, a multi-index will be an ordered s-uple $I = (i_1, \ldots, i_s)$; the length of I is given by the number s; and an expression such as Ij denotes the multi-index given by the $(s+1)$-uple $(i_1, \ldots, i_s, j)$.

As much as the integration by parts procedure is concerned, we will use the local formula $\omega^\sigma_{Ii} \wedge ds = -d\omega_I \wedge ds_i$. We also recall the properties $d_J\omega^\sigma = \omega^\sigma_J$ and $\frac{\partial}{\partial y^\nu_J} \rfloor \omega^\sigma_I = \delta^\sigma_\nu \delta^J_I$ (where the Kronecker symbol with multi-indices has the obvious meaning: it is 1 if the multi-indices coincide up to a rearrangement and 0 otherwise).

Finally, if ψ is a projectable vector field on Y (i.e., an infinitesimal automorphism preserving the fibration), $j^k\psi$ is the projectable vector field, defined on J^kY, associated with the prolongation of the flow of ψ (see, e.g., Refs. 13, 4, and 5).

The contact structure of jet prolongations enables us to define an algebro-geometric object deeply related to the calculus of variations: a differential sequence of sheaves made of equivalence classes of differential forms taking a variational meaning. We refer to Ref. 16, where the construction of a sequence of "variational sheaves" can be found, and to Refs. 15 and 17 for the representation of finite order variational sequences. The concept of a sheaf is due to Leray;[18] a classical reference on this topic is, e.g., Ref. 19.

Let Ω^k_q denote the sheaf of differential q-forms on J^kY. It can be seen as a sheaf on Y; in fact, we assign to an open set $W \subseteq Y$ a form defined on $\pi^{-1}_{r,0}(W)$. We set $\Omega^k_{0,c} = \{0\}$ and denote by $\Omega^k_{q,c}$ the sheaf of contact q-forms, for $q \le n$, or the sheaf of strongly contact q-forms if $q > n$. The quotient sequence of the de Rham sequence of forms,

$$\{0\} \to \mathbb{R}_Y \to \Omega^k_0 \to \cdots \to \Omega^k_n/\Theta^k_n \to \Omega^k_{n+1}/\Theta^k_{n+1} \to \cdots \to \Omega^k_N \to \{0\},$$

where $\Theta^k_q = \Omega^k_{q,c} + d\Omega^k_{q-1,c}$, $N = \dim(J^kY)$, and $\mathbb{R}_Y$ is the constant sheaf over Y, is called *Krupka's variational sequence of order k*.[16] Let us denote the quotient sheaves by $\mathscr{V}^k_q$. Morphisms in this sequence are denoted by $\mathscr{E}_q$: $\mathscr{V}^k_q \to \mathscr{V}^k_{q+1}$, and they are quotients of the exterior differential, i.e., $\mathscr{E}_q([\rho]) = [d\rho]$. By this construction, classes of forms modulo contact forms are interpreted as differential forms relevant for the calculus of variation (Lagrangians, currents, source forms, and so on); the Euler–Lagrange mapping can be identified with a morphism in the variational sequence. The representation of the second variational derivative has been studied from this point of view.[20–23]

The interest of this construction in physics is that the cohomology of the complex of the global section of the variational sequence is the de Rham cohomology of Y.[16] Dealing with the exact sequence of sheaves and resolutions enables us to study cohomology obstructions to *variational exactness of variationally closed forms,* and this turns out to be of interest in many different areas of physics; for example, an obstruction to the existence of global extremals is related to the obstruction to the existence of global Noether–Bessel–Hagen currents.[24]

Strictly related to concrete applications is, then, the so-called *representation problem*, which, roughly speaking, consists in showing that *classes of forms*, i.e., elements of the quotient groups $\mathscr{V}^r_q$, can be associated with *global differential forms*. By the intrinsic geometric structure of the calculus of variations on finite order prolongations of fibrations, indeed, it is possible to define an operator (called *representation mapping*), which takes differential forms on the prolongation of order r and associates with it a differential form on a certain prolongation order $s \ge r$, having a meaning in the Lagrangian formalism for field theory, i.e., $R^r_q : \Omega^r_q \to \Psi^s_q$, with Ψ^s_q being an Abelian group of forms of order s, such that $\ker R^r_q = \Theta^r_q$. It provides an isomorphism $\mathscr{V}^r_q \cong \Psi^s_q = R^r_q(\Omega^r_q)$.

For $q \le n$, R^r_q can be taken to be simply the "horizontalization" $h = p_0$. For $q \ge n+1$, it is the image of an operator denoted by $\mathscr{I}$, which will be suitably defined below and which reflects in an intrinsic way the procedure of getting a distinguished representative of a class $[\rho] \in \Omega^k_q/\Theta^k_q$ for $q > n$ by applying to ρ the operator p_{q-n} and then factorizing by Θ^k_q; see, e.g., Ref. 17.

In this paper, we will refer to the *interior Euler operator* defined within the finite order variational sequence according to Refs. 25 and 15 and applied to the representation of variational Lie derivatives according to Ref. 17.

Definition II.1. In the following, differential forms that are ω^σ generated l-contact $(n+l)$-forms will be called *source forms*.

Now, define locally the map $\mathscr{I} : \Omega^r_{n+k} \to \Omega^{2r+1}_{n+k}$ by

$$\mathscr{I}(\rho) = \frac{1}{k}\omega^\sigma \wedge I_\sigma = \frac{1}{k}\omega^\sigma \wedge \sum_{|I|=0}^{r} (-1)^{|I|} d_I\left(\frac{\partial}{\partial y^\sigma_I} \rfloor p_k\rho\right).$$

For a given ρ, $\mathscr{I}(\rho)$ is a source form of degree $n+k$, and it is by construction a k-contact form. It turns out that if ρ is global, $\mathscr{I}(\rho)$ is a globally defined form; for a proof, see Ref. 15.

In view of a characterization of Noether currents, we study the difference between $\mathscr{I}(\rho)$ and $(\pi_{2r+1,r+1})^*(p_k\rho)$. In particular, we define the *residual operator* $\mathscr{R}$ by the decomposition formula, which is, in fact, a *geometric integration parts formula,*

$$(\pi_{2r+1,r+1})^*(p_k\rho) = \mathscr{I}(\rho) + p_k dp_k\mathscr{R}(\rho). \tag{1}$$

Note that although the decomposition mentioned above has a global meaning, $\mathscr{R}(\rho)$ is a strongly contact $(n+k-1)$-form defined only locally.

Example II.2. Following Ref. 15, we can characterize $\mathscr{R}(\rho)$ in local coordinates. For $k \geq 1$, if Ψ^I_σ are $(k-1)$-contact $(n+k-1)$-forms and if ω^σ_I are local generators of contact one-forms, up to pull-backs, we can write (a sort of integration by parts on formal derivatives of forms)

$$p_k\rho = \sum_{|I|=0}^{r} \omega^\sigma_I \wedge \Psi^I_\sigma = \sum_{|I|=0}^{r} d_I(\omega^\sigma \wedge \zeta^I_\sigma) = \mathscr{I}(\rho) + p_k dp_k \mathscr{R}(\rho),$$

with $\zeta^I_\sigma = \sum_{|J|=0}^{r-|I|} (-1)^{|J|} \binom{|I|+|J|}{|J|} d_J \Psi^{JI}_\sigma$. The first term gives us the Euler–Lagrange form, while by rewriting $\omega^\sigma \wedge \zeta^I_\sigma = \Phi^I \wedge ds$, for suitable k-contact k-forms Φ^I on $J^{2r}Y$, we get

$$\sum_{|I|=1}^{r} d_I(\omega^\sigma \wedge \zeta^I_\sigma) = d_H\Big(\sum_{|I|=0}^{r-1} (-1)^k d_I \Phi^{Ij} \wedge ds_j\Big) = d_H \mathscr{R}(\rho).$$

This local expressions for $\mathscr{R}(\rho)$ will be exploited in Example V.2 for the case $k = 1$, specifically for concrete one-contact $(n+1)$-forms $\omega^\sigma_I \wedge \Psi^I_\sigma$ associated with the exterior differential of a suitably "deformed" Yang–Mills Lagrangian. We will explicitly write the forms Φ^{Ij} relative to this Lagrangian. Combined with results of Corollary IV.15, this approach will enable us to obtain explicit conserved currents associated with symmetry transformations of Yang–Mills extremals on Minkowski space-times.

III. HIGHER VARIATIONS AND RELATED CURRENTS

The representation by the horizontalization h and by the interior Euler operator $\mathscr{I}$ (also called the Takens representation[17]) defines a sequence of sheaves of *differential forms* (rather than of classes of differential forms) such that both the objects and the morphisms have a straightforward interpretation in the calculus of variations. We can obtain formulas for (higher) variations of a Lagrangian based on an iteration of the first variation formula expressed through this representation.

A. Noether currents

The formulation of the first Noether theorem[26] is concerned with the representation of *variational Lie derivatives* of classes of degree n, which illustrates the relation between the interior Euler operator, the Euler–Lagrange operator, and the exterior differential, as well as the emerging of the divergence of the Noether currents by contact decompositions and geometric integration by parts formulas.

In the following, for any n-form ρ, $\mathscr{I}(d\rho) = \mathscr{I}(dh\rho) = \mathscr{I}d(h\rho)$ is the Euler–Lagrange form $E_n(h\rho)$, while $hdh\mu = p_0 dp_0 \mu$ is the horizontal differential $d_H(h\mu)$, which can be recognized as a divergence (for the notation and the interpretation in the context of geometric calculus of variations, more details can be found, e.g., in Ref. 17).

Proposition III.1. For any n-form ρ and for any π-projectable vector field ψ on $\boldsymbol{Y}$, we have, up to pull-backs by projections,

$$L_{J^{r+1}\psi} h\rho = \psi_V \rfloor \mathscr{I} d(h\rho) + d_H(J^{r+1}\psi_V \rfloor p_{d_V h\rho} + \psi_H \rfloor h\rho), \tag{2}$$

where $p_{d_V h\rho} = -p_1 \mathscr{R}(dh\rho)$.

The formula mentioned above was first obtained by Noether in the proof of her celebrated first theorem (see the original paper of Noether in the historical survey in Ref. 27). This suggests the definition of a Noether current.

Definition III.2. The *Noether current for a Lagrangian λ associated with ψ* is defined as

$$\epsilon_\psi(\lambda) = J^{r+1}\psi_V \rfloor p_{d_V\lambda} + \psi_H \rfloor \lambda.$$

The term $p_{d_V\lambda} = -p_1 \mathscr{R}(d\lambda)$ is called *a local generalized momentum*.

It should be stressed that a Noether current is defined for any projectable vector field, independently from it being a Lagrangian symmetry or not. When it is not a symmetry, of course, the Noether current is *not* conserved along critical sections.

A generalization of formula (2) to the class of degree greater or lower than n has been obtained.[8,17,28]

B. Higher Noether currents

We now tackle a systematic formulation of higher variations, interpretable as variations of suitable "deformed" Lagrangians;[6] combined with symmetry considerations, this approach extends to field theory the concept of the so-called higher order Noether symmetries in mechanics developed in Ref. 7.

Now, we obtain a formula for the second variation, which will be further exploited in Sec. IV. We note that $L_{J^{r+1}\psi}h\rho = hL_{J^r\psi}\rho$ and then apply a standard inductive reasoning. Of course, the iterated variation is pulled-back up to a suitable order in order to suitably split the Lie derivatives.[6]

Proposition III.3. For any n-form ρ and any pair of π-projectable vector fields ψ_1 and ψ_2, we have, up to pull-backs by projections,

$$L_{J^{r+1}\psi_2}L_{J^{r+1}\psi_1}h\rho = \psi_{2,V}\rfloor\mathcal{I}d(\psi_{1,V}\rfloor\mathcal{I}d(h\rho))+ \\ + d_H\varepsilon_{\psi_2}(\psi_{1,V}\rfloor\mathcal{I}d(h\rho)) + d_H\varepsilon_{\psi_2}(d_H\varepsilon_{\psi_1}(h\rho)), \tag{3}$$

where we define the following (higher) Noether currents associated with ψ_2 for the respective new Lagrangians:

$$\varepsilon_{\psi_2}(\psi_{1,V}\rfloor\mathcal{I}d(h\rho)) = \psi_{2,H}\rfloor\psi_{1,V}\rfloor\mathcal{I}d(h\rho) + J^{r+1}\psi_{2,V}\rfloor p_{d_V\psi_{1,V}\rfloor\mathcal{I}d(h\rho)}, \tag{4}$$

$$\varepsilon_{\psi_2}(d_H\varepsilon_{\psi_1}(h\rho)) = \psi_{2,H}\rfloor d_H(J^{r+1}\psi_{1,V}\rfloor p_{d_Vh\rho} + \psi_{1,H}\rfloor h\rho)+, \\ + J^{r+1}\psi_{2,V}\rfloor p_{d_Vd_H(J^{r+1}\psi_{1,V}\rfloor p_{d_Vh\rho}+\psi_{1,H}\rfloor h\rho)}. \tag{5}$$

Note that expression (3) is given in terms of $\mathcal{I}$ and $\mathcal{R}$.

Related to this formula is an identity that will suggest the definition of the Jacobi morphism, with a look to a specific characterization of symmetry transformations of extremals (see Definition IV.6).

Let, then, ψ_1, ψ_2 be *vertical vector fields*. We note that due to the exactness of the representation sequence and by linearity of the Lie derivative (for s a suitable prolongation order), we can write

$$J^s\psi_1\rfloor L_{J^s\psi_2}\mathcal{I}d(h\rho) = \psi_1\rfloor\mathcal{I}d(\psi_2\rfloor\mathcal{I}d(h\rho)) = \\ = L_{J^s\psi_2}L_{J^s\psi_1}h\rho - [\psi_2,\psi_1]\rfloor\mathcal{I}d(h\rho) - d_H\varepsilon_{\psi_2}(d_H\varepsilon_{\psi_1}(h\rho)). \tag{6}$$

From (3), we get then the following identity.

Proposition III.4. For every pair of vertical vector fields ψ_1 and ψ_2, the following holds:

$$\psi_1\rfloor\mathcal{I}d(\psi_2\rfloor\mathcal{I}d(h\rho)) - \psi_2\rfloor\mathcal{I}d(\psi_1\rfloor\mathcal{I}d(h\rho)) = \\ = [\psi_1,\psi_2]\rfloor\mathcal{I}d(h\rho) + d_H(\varepsilon_{\psi_2}(\psi_1\rfloor\mathcal{I}d(h\rho))). \tag{7}$$

Note that, being the vector fields vertical, here, we have $\varepsilon_{\psi_2}(\psi_1\rfloor\mathcal{I}d(h\rho)) = J^{r+1}\psi_2\rfloor p_{d_V\psi_1\rfloor\mathcal{I}d(h\rho)}$. Note also that this current is the Noether current for the "deformed" Lagrangian $\psi_1\rfloor\mathcal{I}d(h\rho)$ and associated with ψ_2.

Remark III.5. As we already mentioned in the Introduction, there exists a concept of a higher order Noether symmetry referring actually to a higher order generalization of a "Noether symmetry" intended as a symmetry of the exterior differential of the Poincaré–Cartan equivalent of a given Lagrangian, i.e., Ψ is called a "Noether symmetry" if $L_\Psi d\theta = 0$. In particular, we refer to the generalization due to Sarlet and Cantrijn,[7] and in the following, we shall call a "Noether symmetry" according to them as a Sarlet–Cantrijn symmetry.

We stress that this symmetry differs from a Noether symmetry as a symmetry of the Lagrangian (due to the fact that the Poincaré–Cartan equivalent of L differs for a one-contact term, let us call it Ω), which is the original meaning also used by Emmy Noether who referred to symmetries of the action, i.e., of the Lagrangian. As well-known symmetries of the Lagrangian are *also* symmetries of the Euler–Lagrange form (which can be expressed in terms of the differential of the Poincaré–Cartan equivalent as $p_1d\theta$), but the converse is not true, in general. Indeed, symmetries of the Euler–Lagrange form are generalized symmetries of the Lagrangian, i.e., symmetries up to a horizontal differential. Therefore, Sarlet–Cantrijn symmetries, which are symmetries of the Poincaré and Cartan equivalent up to a differential, can be identified as a kind of generalized symmetries. Accordingly, higher Noether symmetries and currents in this paper can be compared with Sarlet–Cantrijn ones.

Example III.6. Let us now denote by $\theta = L + \Omega$ the Poincaré–Cartan equivalent of the Lagrangian $L = h\rho$. Let us for a moment skip the prolongation symbols to simplify the notation; we have the following.

Let Ψ denote a projectable vector field, and let $L_\Psi L_\Psi d\theta = d(\Psi\rfloor d(\Psi\rfloor d\theta)) = 0$. On the one hand, by the naturality of the Lie derivative and by Eq. (6), we have

$$L_\Psi L_\Psi d\theta = dL_\Psi L_\Psi\theta = \\ = d[\Psi_V\rfloor\mathcal{I}d(\Psi_V\rfloor\mathcal{I}dL) + d_H\varepsilon_\Psi(\Psi_V\rfloor\mathcal{I}dL) + d_H\varepsilon_\Psi(d_H\varepsilon_\Psi(L)) + L_\Psi L_\Psi\Omega].$$

Let us take the quotient modulo the contact structure Θ, $\mathcal{I}dL_\Psi L_\Psi\theta$. $\mathcal{I}dL_\Psi L_\Psi\Omega = L_\Psi L_\Psi \mathcal{I}d\Omega = 0$, being Ω a contact form and being $d\Omega$ also in the contact sheaf Θ. Thus, we have $\mathcal{I}dL_\Psi L_\Psi\theta = \mathcal{I}d\eta$, with $\eta = \Psi_V \rfloor \mathcal{I}d(\Psi_V\rfloor\mathcal{I}dL) + d_H\varepsilon_\Psi(\Psi_V\rfloor\mathcal{I}dL) + d_H\varepsilon_\Psi(d_H\varepsilon_\Psi(L))$.

Note that under the assumption that $L_\Psi L_\Psi d\theta = 0$, η is a $\mathcal{I}d$-closed form, i.e., $\mathcal{I}d\eta = 0$; then, by the exactness of the variational sequence, we have locally $\Psi_V\rfloor\mathcal{I}d(\Psi_V\rfloor\mathcal{I}dL) = d_H(G - [\varepsilon_\Psi(\Psi_V\rfloor\mathcal{I}dL) + \varepsilon_\Psi(d_H\varepsilon_\Psi(L))])$.

On the other hand, following Ref. 7, let $\beta = \Psi\rfloor d\alpha$, where $\alpha = \Psi\rfloor d\theta$. We have $L_\Psi L_\Psi\theta = \beta + d(\Psi\rfloor d(\Psi\rfloor\theta)) = \beta + d\xi$, and from the above description, we also get $\mathcal{I}d\eta = \mathcal{I}d(\beta + d\xi) = \mathcal{I}d(\beta + d_V\xi)$.

We now elaborate and compare these two issues. Indeed, from Proposition III.3, we have the identity $d_H\varepsilon_{\Psi_V}(\Psi_V\rfloor\mathcal{I}dL) = 0$. Thus,

$$\Psi_V\rfloor\mathcal{I}d(\Psi_V\rfloor\mathcal{I}dL) = d_H(G - \Psi_H\rfloor\Psi_V\rfloor\mathcal{I}dL - \varepsilon_\Psi(d_H\varepsilon_\Psi(L))]).$$

Furthermore, if we assume Ψ_V to be such that $\Psi_V\rfloor\mathcal{I}d(\Psi_V\rfloor\mathcal{I}dL) = 0$ (i.e., to be a Jacobi field; see later), then we get the conservation law,

$$d_H(G - \Psi_H\rfloor\Psi_V\rfloor\mathcal{I}dL - \varepsilon_\Psi(d_H\varepsilon_\Psi(L))) = 0,$$

which along an extremal reads

$$d_H(G - \varepsilon_\Psi(d_H\varepsilon_\Psi(L))) = 0.$$

On the other hand, since $L_\Psi L_\Psi d\theta = d(\Psi\rfloor d(\Psi\rfloor d\theta)) = 0$, then $d\beta = 0$; therefore, locally, $\beta = dF$.[7]

By taking the horizontal part, we have $h\beta = d_HF$. However, from the above description, $d_HG = h\eta = h\beta + d_H\xi$; thus, we can take $G = F + \xi$, and we locally have that, for Ψ_V being a Jacobi field along an extremal,

$$d_HF = d_H(\varepsilon_\Psi(d_H\varepsilon_\Psi(L)) - \xi) = d_H(\varepsilon_\Psi(d_H\varepsilon_\Psi(L)) - \Psi\rfloor d(\Psi\rfloor\theta).$$

Now, up to pull-backs and jet prolongations of the vector field Ψ, we have

$$\begin{aligned}
d_HF = d_H\Big(&\Psi_H\rfloor d_H(\Psi_V\rfloor p_{d_VL} + \Psi_H\rfloor L) + \Psi_V\rfloor p_{d_Vd_H(\Psi_V\rfloor p_{d_VL}+\Psi_H\rfloor L)} + \\
&- \Psi_H\rfloor(d_H(\Psi_V\rfloor p_{d_VL} + \Psi_H\rfloor L)) - \Psi_H\rfloor(d_V(\Psi_V\rfloor p_{d_VL} + \Psi_H\rfloor L)) + \\
&- \Psi_V\rfloor(d_H(\Psi_V\rfloor p_{d_VL} + \Psi_H\rfloor L)) - \Psi_V\rfloor(d_V(\Psi_V\rfloor p_{d_VL} + \Psi_H\rfloor L)))\\
= d_H\Big(&\Psi_V\rfloor p_{d_Vd_H(\Psi_V\rfloor p_{d_VL}+\Psi_H\rfloor L)} - \Psi_H\rfloor(d_V(\Psi_V\rfloor p_{d_VL} + \Psi_H\rfloor L)) + \\
&- \Psi_V\rfloor(d_H(\Psi_V\rfloor p_{d_VL} + \Psi_H\rfloor L)) - \Psi_V\rfloor(d_V(\Psi_V\rfloor p_{d_VL} + \Psi_H\rfloor L))) = \\
= d_H(&\Psi_V\rfloor p_{d_Vd_H(\Psi_V\rfloor p_{d_VL}+\Psi_H\rfloor L)} - \Psi_V\rfloor(d_V(\Psi_V\rfloor p_{d_VL} + \Psi_H\rfloor L))\\
= d_H(&\varepsilon_{\Psi_V}(d_H\varepsilon_{\Psi_V}(L) + \Psi_V\rfloor p_{d_Vd_H(\Psi_H\rfloor L)} - \Psi_V\rfloor(d_V(\Psi_V\rfloor p_{d_VL} + \Psi_H\rfloor L)),
\end{aligned}$$

which explicates the relationship between Sarlet–Cantrijn second order Noether conserved current and the conserved current along extremals $\varepsilon_{\Psi_V}(d_H\varepsilon_{\Psi_V}(L))$, for Ψ_V being a Jacobi field; see later Eq. (14) combined with Proposition III.4. The last term can be shown to vanish under the horizontal differential;[8,17,28] thus, in the case of a *vertical* Sarlet–Cantrijn symmetry, the conserved current F essentially coincides with the Noether conserved current $\varepsilon_{\Psi_V}(L_{\Psi_V}(h\rho))$ up to a horizontal differential.

By Propositions III.1 and III.3, formulas for the higher variations of $h\rho$ are obtained as an original result.

Proposition III.7. Let ρ be an n-form on J^kY. Consider the Lagrangian $h\rho$, and take l variation vector fields $\psi_1, \ldots, \psi_l$. Define recursively a sequence r_l by

$$r_l = 2r_{l-1} + 1, \quad r_0 = r.$$

We have

$$\begin{aligned}
(\pi_{r_l,r+1})^*(L_{J^{r+1}\psi_l}\ \ldots L_{J^{r+1}\psi_1}h\rho)&\\
= \psi_{l,V}\rfloor\mathcal{I}d(\psi_{l-1,V}\rfloor\mathcal{I}d(\ldots\psi_{2,V}\rfloor\mathcal{I}d(\psi_{1,V}\rfloor\mathcal{I}d(h\rho))\ldots)+&\\
+ d_H\varepsilon_{\psi_l}(\psi_{l-1,V}\rfloor\mathcal{I}d(\ldots\psi_{2,V}\rfloor\mathcal{I}d(\psi_{1,V}\rfloor\mathcal{I}d(h\rho))\ldots)+&\\
+ d_H\varepsilon_{\psi_l}(d_H\varepsilon_{\psi_{l-1}}(\psi_{l-2,V}\rfloor\mathcal{I}d(\ldots(\psi_{1,V}\rfloor\mathcal{I}d(h\rho))\ldots)+&\\
\ldots&\\
+ d_H\varepsilon_{\psi_l}(d_H\varepsilon_{\psi_{l-1}}d_H(\ldots d_H\varepsilon_{\psi_2}(d_H\varepsilon_{\psi_1}(h\rho))\ldots).&
\end{aligned} \tag{8}$$

Proof. The proof is a straightforward induction using as base step the case $l = 1$ or $l = 2$. Taking into account the exactness of the representation sequence, the inductive step follows easily, although not trivially; details of the proof can be found in Ref. 29.

Remark III.8. By means of a recursive application of (2) or (3), the currents that appear in formula (8) can be worked out more explicitly and characterized as Noether currents similarly to the expressions in (4) and (5).

The variation of any order of a Lagrangian $h\rho$ is a horizontal form, i.e., again a Lagrangian; therefore, we can express its variation by means of formula (2). On the other hand, formula (8) gives us the possibility to investigate how to relate the *symmetries* of a variation of $h\rho$ to $h\rho$ itself (see Theorem IV.11).

IV. SYMMETRY TRANSFORMATIONS OF EXTREMALS AND CONSERVED CURRENTS

Let $h\rho$ be a Lagrangian of order $r + 1$ on Y.

Definition IV.1. A (local) section γ is an extremal of $h\rho$ if it satisfies

$$\mathcal{I} d(h\rho) \circ J^{2r+1}\gamma = 0.$$

Let now ϕ be an automorphism of Y (i.e., a transformation preserving the fibration) with projection ϕ_0, and let $J^{r+1}\phi$ be its prolongation.

Definition IV.2. The automorphism ϕ is a symmetry transformation of an extremal γ if the section $\phi \circ \gamma \circ \phi_0^{-1}$ is also an extremal, i.e.,

$$\mathcal{I} d(h\rho) \circ J^{2r+1}(\phi \circ \gamma \circ \phi_0^{-1}) = 0.$$

A π-projectable vector field ψ is the generator of symmetry transformations of γ if its local one-parameter group of transformations is a flow of symmetry transformations of γ. It can be shown that a symmetry of $\mathcal{I} d(h\rho)$ is also a symmetry transformation of every extremal γ; see Refs. 13 and 5.

According to the above references, the following relates symmetry transformations of extremals to projectable vector fields dragging $h\rho$ in such a way that $L_{J^{r+1}\psi} h\rho$ admits the same extremals; for the proof, see, in particular, Ref. 14.

Theorem IV.3. *Let $h\rho$ be a Lagrangian of order $r + 1$, and let γ be an extremal. Then, a π-projectable vector field ψ generates symmetry transformations of γ if and only if*

$$\mathcal{I} d(L_{J^{r+1}\psi} h\rho) \circ J^{2r+1}\gamma = 0.$$

Remark IV.4. We make now an observation that will have a fundamental consequence when related to Proposition III.2. Indeed, we note that, being the Lie derivative a natural operator, it holds $L_{J^{2r+1}\psi}\mathcal{I} d(h\rho) = \mathcal{I} d(L_{J^{r+1}\psi} h\rho)$, and ψ generates symmetry transformations of γ if and only if

$$(L_{J^{2r+1}\psi}\mathcal{I} d(h\rho)) \circ J^{2r+1}\gamma = 0.$$

We thus characterize vertical symmetry transformations of extremals as particular transformations of the Euler–Lagrange forms to source forms vanishing along extremals of the original Lagrangian.

We are now able to state our *first main result*, which is the premise for the next fundamental step: to characterize vertical symmetry transformations of extremals specifically as Jacobi fields along extremals (see Theorem IV.7 and Remark IV.14). We do this basically by expressing the Lie derivative of Euler–Lagrange forms in terms of the second variation (see also Ref. 22).

Theorem IV.5. *Let $h\rho$ be a Lagrangian of order $r + 1$, and let γ be an extremal. Then, a vertical vector field ψ generates vertical symmetry transformations of γ if and only if*

$$\mathcal{I} d(\psi \rfloor \mathcal{I} d(h\rho)) \circ J^{4r+1}\gamma = 0.$$

Proof. The result follows from Theorem IV.3 and Remark IV.4 by using Proposition III.2 via identity (6), which, we stress, holds true for any vertical vector field ψ_1.

We focus on higher order Noether currents and, in particular, on currents associated with the infinitesimal second variation formula (3) in a specific way.

Roughly speaking, *up to horizontal differentials*, the second variation (generated by vertical vector fields) of a Lagrangian λ is the Jacobi morphism (see Ref. 3 for first order field theory; see also Ref. 21). Here, we define the Jacobi morphism within the representation sequence, i.e., by the interior Euler operator.

Definition IV.6. Let $X_V(Y)$ be the space of vertical vector fields on Y. The map

$$\mathscr{J} : \Omega^r_{n,X}(J^rY) \to X^*_V(J^{2r+1}Y) \otimes X^*_V(Y) \otimes \Omega^r_{n,X}(J^rY), \tag{9}$$

$$\lambda :\mapsto \bullet \rfloor \mathscr{I} d(\bullet \rfloor \mathscr{I} d(\lambda)) \tag{10}$$

is called the Jacobi morphism associated with the Lagrangian λ.

The Jacobi morphism is self-adjoint along critical sections of a Lagrangian field theory *of any order*. This is a property of great importance in physical applications, and we recall its full statement for the role it plays in the paper.

Theorem IV.7. *For any pair of vertical vector fields ψ_1, ψ_2 on Y, we have*

$$J^{2r+1}\psi_2 \rfloor \mathscr{I}(J^{2r+1}\psi_1 \rfloor d\mathscr{I}(d\lambda)) = 0.$$

Along extremals, the Jacobi morphism is self-adjoint.

Indeed, we have

$$\mathscr{I} d(\psi \rfloor \mathscr{I} d(\lambda)) = \sum_{|J|=0}^{2r+1} (-1)^{|J|} d_J (\psi^\rho \frac{\partial E_\rho(\lambda)}{\partial y^\sigma_J}) \omega^\sigma \wedge ds = \tag{11}$$

$$= \sum_{|J|=0}^{2r+1} d_J \psi^\sigma \frac{\partial E_\rho(\lambda)}{\partial y^\sigma_J} \omega^\rho \wedge ds. \tag{12}$$

For the full proof, extending a Goldschmidt and Sternberg result for first order theories,[3] see Ref. 6, as well as, in a slightly different context, Ref. 21.

In the following, we use the notation $\mathscr{J}_\psi(\lambda)$ for short to denote $\mathscr{I} d(\psi \rfloor \mathscr{I} d(\lambda))$.

Definition IV.8. Let λ be a Lagrangian of order r. A *Jacobi field* for the Lagrangian λ is a vertical vector field ψ that belongs to the kernel of the Jacobi morphism, i.e., satisfying the *Jacobi equation* for the Lagrangian λ,

$$\mathscr{J}_\psi(\lambda) = 0.$$

Remark IV.9. The Jacobi morphism $\mathscr{J}_\psi(\lambda)$, evaluated along an extremal γ, depends only on the values of the vector field ψ along γ.

A Jacobi equation along an extremal is then well defined; we call its solutions the *Jacobi fields along an extremal γ.*

Remark IV.10. Equation (12) provides the "adjoint expression" for the Jacobi equation along extremals; it can be of use in order to obtain an easier characterization of the kernel of the Jacobi morphism in practical computations.

Notably, here, it will be used in order to calculate the conserved current associated with invariant sets of Yang–Mills connections (see *Example V.2*). See also Ref. 30 for an explicit application in $SU(3)$-Yang–Mills theories in the context of a variationally featured symmetry breaking in view of a canonical characterization of confinement phases in non-Abelian gauge theories.[31]

A. Jacobi fields and higher conservation laws

We observe that the Jacobi equation for variations of $h\rho$ can be expressed in terms of $h\rho$. In fact, just using the exactness of the representation sequence, we have

$$\mathscr{I} d(\psi \rfloor \mathscr{I} d(\rfloor L_{J^{r+1}\psi_s} \ldots L_{J^{r+1}\psi_1} h\rho)) = \mathscr{I} d(\psi \rfloor \mathscr{I} d(\psi_s \rfloor \mathscr{I} d(\ldots \psi_1 \rfloor \mathscr{I} d(h\rho) \ldots))).$$

The application of Proposition III.6 to an iterated variation of a Lagrangian gives results that are relevant for the Lagrangian itself; in fact, using (8), we can relate the Noether current of the sth variation to Noether currents of lower order variations. More precisely, we can state the following important *original* result.

Theorem IV.11. *If we take a symmetry of an $(l-1)$-th variation of $h\rho$ and we suppose that the sth variation $(s < l)$ is taken with respect to a Jacobi field of the $(s-1)$-th variation, then*

$$d_H\varepsilon_{\psi_l}\dots d_H\varepsilon_{\psi_{s+1}}(L_{J^{r+1}\psi_s}\dots L_{J^{r+1}\psi_1}h\rho)=0.$$

Proof. Actually, we have

$$d_H\varepsilon_{\psi_l}(L_{J^{r+1}\psi_{l-1}}\dots L_{J^{r+1}\psi_1}h\rho)=0,$$

with some terms that vanish separately. In fact, applying the definition of the Jacobi field, we get

$$\begin{aligned}\psi_l\rfloor\mathcal{J}d(\,\psi_{l-1}\rfloor\mathcal{J}d(\dots\psi_2\rfloor\mathcal{J}d(\psi_1\rfloor\mathcal{J}d(h\rho))\dots)&=0,\\ d_H\varepsilon_{\psi_l}(\,\psi_{l-1}\rfloor\mathcal{J}d(\dots\psi_2\rfloor\mathcal{J}d(\psi_1\rfloor\mathcal{J}d(h\rho))\dots)&=0,\\ &\dots\\ d_H\varepsilon_{\psi_l}\dots d_H\varepsilon_{\psi_{s+2}}(\psi_{s+1}\rfloor\mathcal{J}d(\dots(\psi_1\rfloor\mathcal{J}d(h\rho))\dots)&=0.\end{aligned}$$

Then, the statement follows by using (8) (Proposition III.7).

Remark IV.12. The previous result is a strong conservation law (i.e., it holds along any section, not necessarily an extremal): the conserved current is the $(l-s-1)$-th variation of the horizontal differential of the Noether current for the sth variation of $h\rho$. The result is not trivial because we are not assuming that ψ_{s+1} is a symmetry of the sth variation.

Remark IV.13. We can write formula (8) in terms of Jacobi morphisms,

$$\begin{aligned}&(\pi_{r_l,r+1})^*\big(L_{J^{r+1}\psi_l}\dots L_{J^{r+1}\psi_1}h\rho\big)\\ &\quad=\psi_{l,V}\rfloor\mathcal{J}_{\psi_{l-1,V}}\big(\psi_{l-2,V}\rfloor\mathcal{J}_{\psi_{l-3,V}}\dots(h\rho)\dots\big)+\\ &\quad+d_H\varepsilon_{\psi_l}(\psi_{l-1,V}\rfloor\mathcal{J}_{\psi_{l-2,V}}(\psi_{l-3,V}\rfloor\dots(h\rho)\dots))+\\ &\quad+d_H\varepsilon_{\psi_l}(d_H\varepsilon_{\psi_{l-1}}(\psi_{l-2,V}\rfloor\mathcal{J}_{\psi_{l-3,V}}(\dots(h\rho)\dots)))+\\ &\quad+\dots+d_H\varepsilon_{\psi_l}(d_H\varepsilon_{\psi_{l-1}}d_H(\dots d_H\varepsilon_{\psi_1}(h\rho)\dots)).\end{aligned}$$

Remark IV.14. Note that by Theorem IV.5 and by Theorem IV.7, Jacobi fields along extremals are vertical symmetry transformations of extremals and *vice versa*.

Our characterization of (vertical) symmetry transformations of extremals as Jacobi fields along extremals is motivated by the fact that there can exist conservation laws associated with symmetry transformations, which, in principle, are different from the Noether or the Noether–Bessel–Hagen conservation laws associated with symmetries or generalized symmetries of a Lagrangian λ; in particular, we stress once more that *all symmetries of equations are also symmetry transformations of extremals, but the converse is not true, in general*.

Indeed, we have the following as our *further main result*.

Corollary IV.15. Let ρ be an n-form on $J^{r-1}Y$ and $h\rho$ be the associated Lagrangian on J^rY. Let ψ_1 and ψ_2 on Y be two generators of vertical symmetry transformations of extremals. Then, along extremals of $h\rho$, the weak conservation law holds true,

$$d_H\epsilon_{\psi_2}(\psi_1\rfloor\mathcal{J}d(h\rho))=0. \tag{13}$$

Proof. Indeed, by Theorem IV.5, the two generators of symmetry transformations ψ_1 and ψ_2 are also Jacobi fields, i.e., they must satisfy $\mathcal{J}_{\psi_i}(h\rho)=0$ for $i=1,2$. Therefore, from (7), since also $[\psi_2,\psi_1]\rfloor\mathcal{J}d(h\rho)$ vanishes along extremals, we get the result.

For the interpretation of this current as a Noether current for a "deformed" Lagrangian, see the note at the end of Proposition III.3.

Remark IV.16. Suppose that ψ_2 is a symmetry of the first variation of $h\rho$ generated by ψ_1 and that ψ_1 and ψ_2 satisfy $\psi_2\rfloor\mathcal{J}_{\psi_1}(h\rho)=0$, then we have a strong conservation law,

$$d_H\epsilon_{\psi_2}(L_{J^r\psi_1}h\rho)=0. \tag{14}$$

Now, it is clear that *along extremals*, taking two vertical symmetry transformations ψ_1 and ψ_2, we get two separated weak conservation laws; see also Ref. 6.

V. CONSERVED CURRENTS FOR INVARIANT YANG–MILLS CONNECTIONS

Let us consider a Yang–Mills theory[32] on the bundle (C_P, π, M) of principal connections with structure bundle (P, p, M, G), G being a *semi-simple group*. Lower Greek indices label space–time coordinates, while capital Latin indices label the Lie algebra $\mathfrak{g}$ of G; then, on the bundle C_P, we introduce coordinates (x^μ, ω^A_σ).

Let δ be the Cartan–Killing metric on the Lie algebra $\mathfrak{g}$, and choose a δ-orthonormal basis T_A in $\mathfrak{g}$; the components of δ will be denoted as δ_{AB}. The Yang–Mills Lagrangian is locally expressed by

$$\lambda_{YM} = -\frac{1}{4} F^A_{\mu\nu} g^{\mu\rho} g^{\nu\sigma} F^B_{\rho\sigma} \delta_{AB} \sqrt{g} ds,$$

where g stands for the absolute value of the determinant of the metric $g_{\mu\nu}$, c^A_{BC} are the *structure constants* of $\mathfrak{g}$, $F^A_{\mu\nu} = \omega^A_{\nu,\mu} - \omega^A_{\mu,\nu} + c^A_{BC}\omega^B_\mu \omega^C_\nu$ is the so-called *field strength*, and we set $\omega^A_{\mu,\nu} = d_\nu \omega^A_\mu$.

From now on, we assume the metric η to be Minkowskian; in this case, the Euler–Lagrange expressions for the Yang–Mills Lagrangian are explicitly written as

$$\begin{aligned} E^\nu_B = \delta_{BA}\eta^{\lambda\mu}\eta^{\varepsilon\nu}(\omega^A_{\varepsilon,\lambda\mu} - \omega^A_{\lambda,\varepsilon\mu} + c^A_{ZD}\omega^Z_{\lambda,\mu}\omega^D_\varepsilon + c^A_{ZD}\omega^Z_\lambda \omega^D_{\varepsilon,\mu}) + \\ + \eta^{\lambda\mu}\eta^{\varepsilon\nu}\delta_{DA}(\omega^D_{\varepsilon,\lambda} - \omega^D_{\lambda,\varepsilon} + c^D_{EF}\omega^E_\lambda \omega^F_\varepsilon) c^A_{BZ}\omega^Z_\mu. \end{aligned} \tag{15}$$

Let (ϕ^a) be a set of coordinates on the group G. A vertical vector field over C_P has the form $\psi = \psi^Z_\sigma \frac{\partial}{\partial \omega^Z_\sigma}$, and its components satisfy the transformation rule $\psi'^B_\nu = Ad^B_A(\phi)\psi^A_\mu \bar{J}^\mu_\nu$, where $Ad^B_A(\phi)$ is the adjoint representation of G on $\mathfrak{g}$ and $\bar{J}^\mu_\nu$ denotes the inverse of the matrix of the change in coordinates in the base space. Let $L(M)$ be the frame bundle of M, and $V = \mathfrak{g} \otimes \mathbb{R}^n$. Let us denote by ∇ the covariant derivative corresponding to $\Omega = dx^\mu \otimes (\partial_\mu + c^B_{AD}\psi^D_\sigma \omega^A_\mu \partial^\sigma_B)$, the connection induced on the bundle $(P \times_M L(M)) \times_\lambda V$ of vertical vector fields over C_P, where the representation λ comes from the transformation rules mentioned above (see Refs. 33, 34, and 2).

By some careful manipulations (see Ref. 6 for details), Theorem IV.7 and specifically formula (12) provide *the Jacobi equation along extremals* for this kind of Yang–Mills theory. In particular, due to the antisymmetry of $F^D_{\beta\sigma}$ in the lower indices, it splits into the antisymmetric and symmetric parts

$$\begin{aligned} \eta^{\nu[\sigma}\eta^{\beta]\alpha}\Big\{\nabla_\beta\big[\big(\nabla_\alpha\psi^A_\sigma - \nabla_\sigma\psi^A_\alpha\big)\delta_{BA}\big] + F^D_{\beta\sigma}\delta_{AD}c^A_{BZ}\psi^Z_\alpha\Big\} = 0, \\ \eta^{\nu(\sigma}\eta^{\beta)\alpha}\big\{\nabla_\beta\big[\big(\nabla_\alpha\psi^A_\sigma - \nabla_\sigma\psi^A_\alpha\big)\delta_{BA}\big]\big\} = 0 \end{aligned} \tag{16}$$

for any pair (ν, B); hereafter, the brackets () and [] in the superscripts denote symmetrization and anti-symmetrization, respectively.

Note that the left-hand side of these equations is the analogous, for a Minkowskian metric, of the classical expression for the Jacobi operator for Yang–Mills theories on different backgrounds (see, e.g., Refs. 35 and 37), and it reproduces results for first order non-regular Lagrangians.[3]

In order to avoid notational confusion, let $\chi^A_\mu, \chi^A_{\mu,\nu}, \chi^A_{\mu,\nu\rho}, \ldots$ denote generators of contact forms.

Remark V.1. According to our results, the solutions ψ of the above equations are the generators of symmetry transformations of Yang–Mills extremals ω, i.e., if ψ is a solution of the above equation, then the source form $L_{J^3\psi}(E^\nu_B \chi^B_\nu \wedge ds)$, where E^ν_B are given by (15), also vanishes along the same extremals, i.e.,

$$(L_{J^3\psi}(E^\nu_B\ \chi^B_\nu \wedge ds)) \circ J^3\omega = 0.$$

Here, by a slight abuse of notation, we denoted by ω a section of the bundle (C_P, π, M), which is an extremal, i.e., a Yang–Mills connection.

As we already mentioned, compared with transformations leaving invariant the Euler–Lagrange form $E^\nu_B \chi^B_\nu \wedge ds$, such transformations are involved with a weaker invariance property since the Euler–Lagrange form is not invariant under their action, but it is transformed to a source form having the same extremals. Note that, indeed, the Yang–Mills extremals ω are also solutions of the equation mentioned above *and vice versa*.

Example V.2. As an illustration of the application of our main result, given by Theorem IV.5 and Corollary IV.15, we determine the conserved current associated with such a weaker invariance property. We write down explicitly the current for two given generators of vertical symmetry transformations ψ and $\tilde{\psi}$, solutions of Eq. (16) (for details, we refer to Refs. 29 and 6, where computations are made for Jacobi fields along extremals).

Being the vector fields vertical, from Corollary IV.15, Eq. (13), and according to Proposition III.1, the conserved current along an extremal is defined through $p_1\mathscr{R}\,(\,d(\psi \rfloor \mathscr{I} d(\lambda_{YM}))$. Recalling that E^ν_B are coordinate expressions of the Euler–Lagrange form according to (15), we apply the coordinate characterization of the residual operator (given in Example II.2) to the following form:

$$d(\psi\rfloor\mathscr{I}d(\lambda_{YM}))=\left(\frac{\partial\psi^B_\nu}{\partial\omega^Z_\rho}E^\nu_B+\psi^B_\nu\frac{\partial E^\nu_B}{\partial\omega^Z_\rho}\right)\chi^Z_\rho\wedge ds+$$
$$+\left(\psi^B_\nu\frac{\partial E^\nu_B}{\partial\omega^Z_{\rho,\xi}}\right)\chi^Z_{\rho,\xi}\wedge ds+\left(\psi^B_\nu\frac{\partial E^\nu_B}{\partial\omega^Z_{\rho,\xi\tau}}\right)\chi^Z_{\rho,\xi\tau}\wedge ds.$$

By suitably rewriting the above-mentioned equation in the form $\sum_{|I|=0}^{2} d_I(\chi^A_\mu\wedge\zeta^{\mu,I}_A)$, we can easily obtain the local expression for $\mathscr{R}(d(\psi\rfloor\mathscr{I}d(\lambda_{YM})))$; thus, obtaining the conserved current, we are looking for

$$\epsilon_{\tilde{\psi}}(\psi\rfloor\mathscr{I}d(\lambda_{YM}))=\Big[\eta^{\rho[\xi}\eta^{\sigma]\nu}\delta_{BA}c^A_{ZD}\omega^D_\sigma\left(\psi^B_\nu\tilde{\psi}^Z_\rho-\psi^Z_\rho\tilde{\psi}^B_\nu\right)$$
$$\left(\eta^{\xi\sigma}\eta^{\rho\nu}-\eta^{\rho(\sigma}\eta^{\xi)\nu}\right)\left(\psi^Z_\nu\nabla_\sigma\left(\tilde{\psi}^B_\rho\delta_{BZ}\right)-\tilde{\psi}^Z_\rho\nabla_\sigma\left(\psi^B_\nu\delta_{BZ}\right)\right)\Big]ds_\xi.$$

Remark V.3. It is noteworthy that, by Proposition III.3 and, in particular, by Remark IV.14, here, the existence and the meaning of the above-mentioned current (also appeared in Ref. 6 in relation to Jacobi fields) is understood under a new light, definitely *relevant from a physical point of view*.

In the present paper, we clarify that such a conservation law emerges by an invariance property of the set of extremals and, moreover, that the associated conserved current can be interpreted as a very specific kind of Noether current, the existence of which is related to a wide class of symmetry transformations. Indeed, we proved that this current can be identified as the Noether current for the Yang–Mills Lagrangian "deformed" by the symmetry transformation of extremals ψ and associated with (or generated by) the symmetry transformation of extremals $\tilde{\psi}$.

Remark V.4. We note that Eq. (7) of Proposition III.3 says us that for any vertical vector field $\psi_1=\psi_2=\zeta$, the current $\epsilon_\zeta(\zeta\rfloor\mathscr{I}d(h\rho))$ is a strong conserved current (i.e., conserved "off shell"). However, it can be easily checked that, at least in the specific case of study, for any (vertical) symmetry transformation of extremals $\tilde{\psi}=\psi$, the weak (i.e., "on shell") conserved current reduces to $\eta^{\rho(\sigma}\eta^{\xi)\nu}(\psi^Z_\nu\nabla_\sigma(\psi^B_\rho\delta_{BZ})-\psi^Z_\rho\nabla_\sigma(\psi^B_\nu\delta_{BZ}))ds_\xi$, which vanishes identically because $\eta^{\rho(\sigma}\eta^{\xi)\nu}=\eta^{\nu(\sigma}\eta^{\xi)\rho}$. This holds true for any couple of linearly dependent symmetry transformations.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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