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Nuclear Physics B 960 (2020) 115187

NUCLEAR PHYSICS B

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Particle-like, dyx-coaxial and trix-coaxial Lie algebra structures for a multi-dimensional continuous Toda type system

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Received 7 April 2020; received in revised form 14 September 2020; accepted 20 September 2020
Available online 22 September 2020
Editor: Hubert Saleur

Dedicated to Hartwig in occasion of his eighteenth birthday

Abstract

We prove that with a $(2+1)$-dimensional Toda type system are associated algebraic skeletons which are (compatible assemblings) of particle-like Lie algebras of dyons and triadons type. We obtain trix-coaxial and dyx-coaxial Lie algebra structures for the system from algebraic skeletons of some particular choice for compatible associated absolute parallelisms. In particular, by a first choice of the absolute parallelism, we associate with the $(2+1)$-dimensional Toda type system a trix-coaxial Lie algebra structure made of two (compatible) base triadons constituting a 2-catena. Furthermore, by a second choice of the absolute parallelism, we associate a dyx-coaxial Lie algebra structure made of two (compatible) base dyons, as well as particle-like Lie algebra structures made of single 3-dyons. Some explicit examples of applications such as conservation laws related to special solutions, and an inverse spectral problem are worked out.

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https://doi.org/10.1016/j.nuclphysb.2020.115187

1. Introduction

Toda type systems are nonlinear models which play a role in a variety of physical and, more in general, natural phenomena.

The problem of integrability of nonlinear models has been recognized to be related to their algebraic properties in discrete and continuous, as well as, classical and quantum formulations. Algebraic properties can often be interpreted as the counterpart of the concept of integrability given as of having 'enough' conservation laws to exhaustively describe the underlying field or associated dynamics. Indeed, from an historical point of view, algebraic-geometric approaches are based on the requirement for the existence of conservation laws which emerge from internal symmetries (given in terms of algebraic structures).

In the Seventies, in fact, Wahlquist and Estabrook [41,5] proposed a technique for systematically deriving, from an integrable system, what they called a 'prolongation structure' in terms of a set of 'pseudopotentials' related to the existence of an infinite set of associated conservation laws. They also conjectured that, as a characterizing feature of the integrability property, the structure was 'open' *i.e.* not a set of structure relations of a finite–dimensional Lie group. Since then, 'open' Lie algebras have been extensively studied in order to distinguish them from freely generated infinite-dimensional Lie algebras.

Their interest in the study of integrability is in the fact that Lax pairs of the inverse spectral transform containing an isospectral parameter can be obtained by an homomorphism of the infinite-dimensional open Lie algebra in a finite-dimensional 'closed' Lie algebra. In their approach, conservation laws are written in terms of 'prolongation' forms and integrability is intended as a Frobenius integrability condition for a 'prolonged' ideal of differential forms describing intrinsically the given nonlinear model in the sense of É. Cartan.

Attempting a *description of symmetries in terms of Lie algebras* implies the appearance of an homogeneous space and thus the interpretation of prolongation forms as *Cartan–Ehresmann connections*. It is clear that here the unknowns are both conservation laws and symmetries, and the main point in this is how to realize the form of the conservation laws and thus the explicit expression of the prolongation forms. Different prolongation ideals give rise to both different algebraic structures (symmetries) and corresponding conservation laws. By an inverse procedure based on the intrinsic duality between Lie algebras and differential systems [4,18], open Lie algebraic structures can 'generate' whole families of different nonlinear systems bound by the same internal symmetry structure.

In a series of papers [24–26,28–30], we explicated an algebraic-geometric interpretation of the above mentioned 'prolongation' procedure in terms of *towers with infinitesimal algebraic skeletons* (in the sense of [20]) and we will refer to that framework in this paper. It is noteworthy that slight modification of the internal symmetry properties generates *new* models which can contain possible integrable subcases. For example, activator-substrate systems have been obtained by performing a slight modification of the internal symmetry algebra of twisted reaction-diffusion equations [26].

The structure itself with which the tower forms are postulated can produce open algebraic structures or just Lie algebras. Our aim in this work is to investigate some common features of them and to show the emerging of particle-like Lie algebras structure as symmetry structures of integrable systems (associated with Poisson structures the compatibility of which is worthy of study [6,16]). Indeed, in general, infinite dimensional open Lie algebras are the main object of the search in view of the application of the inverse spectral transform to obtain soliton solutions, Bäcklund transformations and so on; recent examples of applications can be found *e.g.* in [13, 14,21,42–44] and references therein. Although these features are not our prominent task in this paper, an inverse problem will be obtained in Section 3.3.1.

Prolongation forms bringing to finite dimensional Lie algebras (without a spectral parameter) are generally discarded when searching for a Lax pair to be used within the inverse spectral transform.

However, integrable systems, admitting infinite-dimensional prolongation Lie algebras can also admit finite-dimensional Lie algebras, which still can be related to some kind of internal symmetries of the systems themselves and to associated conservation laws, or even to Bäcklund transformations. We refer, in particular, to the paper "More prolongation structures" by C. Hoenselaers [11], which pointed out two important features of the algebraic structures obtained by the method of Wahlquist and Estabrook.

First feature: it can be that the prolongation forms can not always be solved in such a way that one obtains commutators among vector fields depending only on the 'pseudopotentials' coordinates. A typical example is, in fact, the most general prolongation problem associated by such a procedure to equation (1): in [27] the prolongation problem was formally solved by introducing suitable operators of Bessel type, however a prolongation algebra (and then an inverse problem) could not be obtained explicitly.

It is noteworthy that a certain arbitrariness is given by postulating the structure of the tower in the search of a skeleton. Say, to a given equation can be associated different towers with different skeletons; for different choices of the prolongation ideal see also *e.g.* [9]. It was already stressed by Estrabrook himself [4] that the same algebraic structure 'contains' families of equations, associated linear spectral problems and Bäcklund transformations, interrelated by transformations between dependent and independent coordinates.

The motivation of our previous research for the interpretation of prolongation structures as skeletons of towers was the aim of deeply understanding such a feature: *equations, linear problems, Bäcklund transformations, are local coordinates expressions of common intrinsic structures*; this is of help also in practical questions: solutions of systems can be obtained by simpler systems having in common (part of) skeletons. In Section 3.2.1 we show that each one of the two compatible 4-triadon constituting the skeleton given by a trix-coaxial Lie algebra structure generates the same conservation law and related special solutions. This justifies the possible choice of a more restricted (instead of the most general one) form of the tower (then of the algebraic skeleton) still getting 'solutions' (with this term meaning analytical solutions as well as particular conservation laws) of the original equation.

Second feature: even if the prolongation algebra is a finite dimensional (even abelian in his example) Lie algebra, nevertheless there can exist Bäcklund transformations. It is found an exterior differential of genus 3 associated with the prolongation structure of a NLS equation, and it is stressed that we can choose dependent and independent variables in an arbitrary way. We can also lower the genus (in our skeleton formulation this means the choice of different representations ρ or even different vector spaces V) so obtaining a reduced ideal where one of the independent coordinates is turned into a dependent coordinate. This turning a global symmetry in a local one provides Miura type transformations between the modified NLS and another system, the prolongation structure of which is finite dimensional and there is only one nontrivial potential entering a Bäcklund transformation acting on the modified NLS equation.

In few words the Wahlquist-Estabrook method not always produces infinite dimensional open Lie algebras, but it could be that by that 'procedure' we get only a part of an algebraic skeleton. Therefore we can not automatically infer that, being the prolongation structure finite dimensional, then the system is not integrable. The results in [11] are a counterexample, which suggests that we could extend a finite dimensional prolongation structure in order to implement the skeleton structure of an integrable family of nonlinear systems. These aspects are related to Olver's symmetry reduction [23]. For hydrodynamic reductions of multidimensional systems see [7] and, in particular, for hydrodynamic reductions of the heavenly equation, see [8].

Moreover, we guess that the skeleton can be further implemented by a finer structure related to particle-like Lie algebra structure and this is the inspiring idea of our investigations. In this note we show that such (otherwise discarded) symmetries deserve a more careful study. We take a $(2+1)$-dimensional Toda type system as a study case and show that it posses algebraic properties related to the recently introduced concept of *particle-like Lie algebra structures* [39,40].

Vinogradov developed a completely abstract theory of compatibility of Lie algebra structures starting from the corresponding compatibility theory of Poisson structures. Although the mathematical aspects of the theory are quite involved the nice point is that simple criteria of compatibility or non compatibility have been obtained which somehow have a certain grade of automatism.

Furthermore, as for the physical side, Vinogradov speculated that this particle-like structures could be related to the ultimate particle structure of the matter: he noted that since

> 'the symmetry algebra $u(2) = so(3)$ of a nucleon can be assembled in one step from three triadons [...] one might think that this structure of the symmetry reflects the fact that a nucleon is made from three "quarks" '.

This is of course only a speculation, but it also suggests a quite fascinating new perspective on internal symmetries of integrable systems.

2. Internal symmetries of Toda type systems in $(2+1)$ dimensions

Consider the $(2+1)$-dimensional system, a continuous (or long-wave) approximation of a spatially two-dimensional Toda lattice [37]:

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0\,, \tag{1}$$

where $u = u(x, y, z)$ is a real field, x, y, z are real local coordinates (if we want, z playing the rôle of a 'time') and the subscripts mean partial derivatives. It can be seen as the limit for $\gamma \to \infty$ of the more general model

$$u_{xx} + u_{yy} + \left[(1 + u/\gamma)^{\gamma-1}\right]_{zz} = 0$$

covering (for $\gamma \neq 0, 1$) various continuous approximations of lattice models, among them the Fermi-Pasta-Ulam ($\gamma = 3$) [1]. This model is almost *ubiquitus*, it appears in differential geometry; in mathematical and theoretical physics (Newman and Penrose); in the theory of Hamiltonian systems; in general relativity; in the large n limit of the $sl(n)$ Toda lattice; in extended conformal symmetries, and theory of gravitational instantons; in strings theory and statistical mechanics *etc.* (see *e.g.* [3,15,17,31,33]).

It can be seen as the particular case with $d = 1$ of so-called $2d$-dimensional Toda-type systems [35,34] obtained from a 'continuum Lie algebra' by means of a zero curvature representation $u_{w\bar{w}} = K(e^u)$, (in our particular case $w = x + iy$ and K is the differential operator given by $K = \frac{\partial^2}{\partial z^2}$). In particular, it has been studied in the context of symmetry reductions [2,10] and a $(1+1)$-dimensional version in the context of prolongation structures [1]. The $(2+1)$-dimensional system has been associated with a Kač–Moody Lie algebra and related to Saveliev's continuum Lie algebras of particular kind [29].

The Toda system (1) can be put in the complex form

$$\partial_\zeta \partial_{\bar{\zeta}} u = -1/4 \partial_z^2 e^u \,,$$

by the transformations $\zeta = g(\eta)$, $\bar{\zeta} = \bar{g}(\bar{\eta})$, $u = \tilde{u} - \ln(g'\bar{g}')$, where $\zeta = x + iy$, $\bar{\zeta} = x - iy$, $\partial_\zeta = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{\zeta}} = \frac{1}{2}(\partial_x + i\partial_y)$, $g' = g_\eta(\eta)$, $\bar{g}' = g_{\bar{\eta}}(\bar{\eta})$ and $g(\eta)$ is an arbitrary holomorphic function of $\eta = x' + iy'$. A Lax pair for this complex form of the $2D$ Toda equation has been found; see *e.g.* Manakov and Santini [19] and references therein; original references are [12,36], as well as [45].

2.1. Skeletons for the (2+1) Toda system

Let us first recall a few mathematical tools constituting the background for a detailed treatment of which we refer to [28–30] and [20,32].

From one side global properties of partial differential equations such as internal symmetries and invariance properties having an issue in dynamics can be described by mathematical tools which enable us to deal with global properties at large scales, connecting local data to global ones. On the other side transformations of configurations of a system can be globally studied by means of the theory of the action of Lie groups on manifolds. The differential content carried by a Lie group (and its Lie algebra) and by its structure equations provides differential equations.

We observe that two ingredients constitute the nonlinear phenomena: symmetries on the one side (algebraic content) and changes in time and space on the other side (differential content). In particular, to keep account of the 'interaction' of both aspects, we recognize a refined structure of open Lie algebraic structures associated with them: we introduce a notion which generalizes the concept of a homogeneous space, *i.e.* that of an *algebraic skeleton* $\boldsymbol{E} = \mathfrak{g} \oplus \boldsymbol{V}$ on a finite-dimensional vector space $\boldsymbol{V}$, with $\mathfrak{g}$ a possibly infinite dimensional Lie algebra. The further step is introducing a tower with such a skeleton.

An *algebraic skeleton* on a finite-dimensional vector space $\boldsymbol{V}$ is a triple $(\boldsymbol{E}, \boldsymbol{G}, \rho)$, with $\boldsymbol{G}$ a (possibly infinite-dimensional) Lie group, $\boldsymbol{E} = \mathfrak{g} \oplus \boldsymbol{V}$ is a (possibly infinite-dimensional) vector space *not necessarily equipped with a Lie algebra structure*, $\mathfrak{g}$ is the Lie algebra of $\boldsymbol{G}$, and ρ is a representation of $\mathfrak{g}$ on $\boldsymbol{E}$ such that it reduces to the adjoint representation of $\mathfrak{g}$ on itself. The fact that $\boldsymbol{E}$ is not a direct sum of Lie algebras, but an open algebraic structure is fundamental in order to be able to generate whole families of nonlinear differential systems, starting from it.

We now consider a suitably constructed differentiable structure which is somewhat modeled on the skeleton above. Let us introduce a differentiable manifold $\boldsymbol{P}$ on which a Lie group $\boldsymbol{G}$, with Lie algebra $\mathfrak{g}$, acts on the right; $\boldsymbol{P}$ is a principal bundle $\boldsymbol{P} \to \boldsymbol{Z} \simeq \boldsymbol{P}/\boldsymbol{G}$. By construction, we have that $\boldsymbol{Z}$ is a manifold of type $\boldsymbol{V}$, *i.e.* $\forall z \in \boldsymbol{Z}$, $T_z\boldsymbol{Z} \simeq \boldsymbol{V}$.

Suppose we have a way to define a representation ρ of the Lie algebra $\mathfrak{g}$ on $T_z\boldsymbol{Z} \simeq \boldsymbol{V}$, in such a way that it could be possible under certain conditions to find a homomorphism between the open infinite dimensional Lie algebra, constructed by ρ, and a quotient Lie algebra. Let us call $\mathfrak{k}$ the (possibly infinite dimensional) Lie algebra obtained as the direct sum of such a quotient Lie algebra with $\mathfrak{g}$. From the differentiable side, a *tower* $\boldsymbol{P}(\boldsymbol{Z}, \boldsymbol{G})$ *on* $\boldsymbol{Z}$ *with skeleton* $(\boldsymbol{E}, \boldsymbol{G}, \rho)$ is an *absolute parallelism* ω on $\boldsymbol{P}$ valued in $\boldsymbol{E}$, invariant with respect to ρ and reproducing elements of $\mathfrak{g}$ from the fundamental vector fields induced on $\boldsymbol{P}$, *i.e.* $R_g^*\omega = \rho(g)^{-1}\omega$, for $g \in \boldsymbol{G}$; $\omega(\tilde{A}) = A$, for $A \in \mathfrak{g}$; here R_g denotes the right translation and $\tilde{A}$ the fundamental vector field induced on $\boldsymbol{P}$ from A. In general, the absolute parallelism *does not* define a Lie algebra homomorphism.

Let then $\mathfrak{k}$ be a Lie algebra and $\mathfrak{g}$ a Lie subalgebra of $\mathfrak{k}$. Let $\boldsymbol{G}$ be a Lie group with Lie algebra $\mathfrak{g}$ and $\boldsymbol{P}(\boldsymbol{Z}, \boldsymbol{G})$ be a principal fiber bundle with structure group $\boldsymbol{G}$ over a manifold $\boldsymbol{Z}$ as above. A *Cartan connection* in $\boldsymbol{P}$ of type $(\mathfrak{k}, \boldsymbol{G})$ is a 1-form ω on $\boldsymbol{P}$ with values in $\mathfrak{k}$ such that $\omega|_{T_{\boldsymbol{p}}\boldsymbol{P}} : T_{\boldsymbol{p}}\boldsymbol{P} \to \mathfrak{k}$ is an isomorphism $\forall \boldsymbol{p} \in \boldsymbol{P}$, $R^*_g \omega = Ad(g)^{-1}\omega$ for $g \in \boldsymbol{G}$ and reproducing elements of $\mathfrak{g}$ from the fundamental vector fields induced on $\boldsymbol{P}$. It is clear that a Cartan connection $(\boldsymbol{P}, \boldsymbol{Z}, \boldsymbol{G}, \omega)$ of type $(\mathfrak{k}, \boldsymbol{G})$ is a special case of a tower on $\boldsymbol{Z}$.

The vector space $\boldsymbol{V}$ is finite dimensional and generated by some of the vector fields in the prolongation structure. It has the property that each bracket of some of remaining vector fields of the prolongation structure (freely generating an infinite dimensional Lie algebra $\mathfrak{g}$) with its generators is again in $\boldsymbol{V}$. In particular unknown commutators in the freely generated Lie algebra are related in such a way that their assigned relations are elements of $\boldsymbol{V}$.

As an example of application of such an abstract formulation to the real world we refer *e.g.* to [26], whereby activator-substrate systems have been obtained by performing a slight modification of the internal symmetry algebra of twisted reaction-diffusion equations: the necessary condition for the generation of stable patterns (related to general integrability properties in the limit of a null normalized diffusion constant) is formulated in terms of 'closeness' properties within the symmetry algebra vector space.

Following [29], we recall how to get both some skeletons and towers over them associated with the system (1).

On a manifold with local coordinates (x, y, z, u, p, q, r), we introduce the closed differential ideal defined by the set of 3-forms: $\theta_1 = du \wedge dx \wedge dy - r dx \wedge dy \wedge dz$, $\theta_2 = du \wedge dy \wedge dz - p dx \wedge dy \wedge dz$, $\theta_3 = du \wedge dx \wedge dz + q dx \wedge dy \wedge dz$, $\theta_4 = dp \wedge dy \wedge dz - dq \wedge dx \wedge dz + e^u dr \wedge dx \wedge dy + e^u r^2 dx \wedge dy \wedge dz$. It is easy to verify that on every integral submanifold defined by $u = u(x, y, z)$, $p = u_x$, $q = u_y$, $r = u_z$, with $dx \wedge dy \wedge dz \neq 0$, the above ideal is equivalent to the Toda system under study.

By an ansatz first introduced in [28], we look for suitable 2-forms (generating associated conservation laws)

$$\Omega^k = \theta^k_m \wedge \omega^m$$

where $\theta^k_m = -\hat{A}^k_m dx - \hat{B}^k_m dy - \hat{C}^k_m dz$, with $\hat{A}^k_m$, $\hat{B}^k_m$, $\hat{C}^k_m$ elements of $N \times N$ constant regular matrices, and the absolute parallelism forms are given by

$$\omega^m = d\hat{\xi}^m + \hat{F}^m dx + \hat{G}^m dy + \hat{H}^m dz \,, \tag{2}$$

i.e.

$$\begin{aligned} \Omega^k = H^k(u, u_x, u_y, u_z; \xi^m) dx \wedge dy + F^k(u, u_x, u_y, u_z; \xi^m) dx \wedge dz + \\ + G^k(u, u_x, u_y, u_z; \xi^m) dy \wedge dz + A^k_m d\xi^m \wedge dx + B^k_m d\xi^m \wedge dz + d\xi^k \wedge dy \,, \end{aligned} \tag{3}$$

where $\xi = \{\xi^m\}$, $k, m = 1, 2, \dots,$ N (N arbitrary), and H^k, F^k and G^k are, respectively, the pseudopotentials and functions to be determined, while A^k_m and B^k_m denote the elements of two $N \times N$ constant regular matrices related to the previous ones and we have rescaled the coordinates ξ^k. In particular note that (see also [28,24])

$$F^k = \hat{C}^k_m \hat{F}^m - \hat{A}^k_m \hat{H}^m \,, \tag{4}$$

$$G^k = \hat{C}^k_m \hat{G}^m - \hat{B}^k_m \hat{H}^m \,, \tag{5}$$

$$H^k = \hat{B}^k_m \hat{F}^m - \hat{A}^k_m \hat{G}^m \,, \tag{6}$$

$$\xi^k = \hat{C}^k_m \hat{\xi}^m \,. \tag{7}$$

The integrability condition for the ideal generated by forms θ_j and Ω^k finally yields

$$H^k = e^u u_z L^k(\xi^m) + P^k(u, \xi^m), \tag{8}$$

$$F^k = -u_y L^k(\xi^m) + Q^k(u, \xi^m), \tag{9}$$

$$G^k = u_x L^k(\xi^m) + M^k(u, \xi^m), \tag{10}$$

where L^k, P^k, Q^k, M^k are functions of integration.

It turns out that $Q^k(u, \xi^m)$ can be written in terms of the others. Indeed we have (see *e.g.* [22,38]) $H^k = B^k_m F^m - A^k_m G^m$ so that $e^u u_z L^k(\xi^l) + P^k(u, \xi^l) = -A^k_m(u_x L^m(\xi^l) + M^m(u, \xi^l)) + B^k_m(-u_y L^m(\xi^l) + Q^m(u, \xi^l))$, *i.e.* $B^k_m Q^m_u(u, \xi^l) = e^u u_z L^k(\xi^l) + P^k_u(u, \xi^l) + A^k_m M^m_u(u, \xi^l)$, which can be integrated once the dependence on of $P^k(u, \xi^l)$ and $M^k(u, \xi^l)$ on u is given. As a consequence, the desired representation ρ for the skeleton is provided by the following equations (we omit the indices for simplicity) [27,29].

$$P_u = e^u [L, M], \quad M_u = -[L, P], \quad [M, P] = 0. \tag{11}$$

Note that here L depends only on ξ^m, while P and M still have a dependence on u determined by the first two differential equations. A tower with P and M given in terms of L has been obtained by suitable *operator Bessel coefficients* [27].

Note that formally this tower shall provide the Lax pair of an inverse spectral problem; however, it is a non trivial task to characterize explicitly its algebraic skeleton by means of the representation provided by the relations $[M, P] = 0$, *i.e.* to obtain a spectral problem in a manageable form.

Particular choices for the absolute parallelism can provide us explicit representations of the prolongation skeleton; in particular a Kač–Moody Lie algebra has been obtained [29] (see Proposition 3.5, case 1. (b) below). In the following we will concentrate on those choices that generate particle-like Lie algebra structures. We shall see that it is yet possible to obtain a spectral problem with a particular choice of the tower.

3. Particle-like Lie algebra structure

Recently, Vinogradov proved that any Lie algebra over an algebraically closed field or over $\mathbb{R}$ can be assembled in a number of steps from two elementary constituents, that he called dyons and triadons [39]. He considered the problems of the construction and classification of those Lie algebras which can be assembled in one step from base dyons and triadons, called coaxial Lie algebras. The base dyons and triadons are Lie algebra structures that have only one non-trivial structure constant in a given basis, while coaxial Lie algebras are linear combinations of pairwise *compatible* base dyons and triadons [40]. Here for the convenience of the reader we recall some basic facts of the theory in the original Vnogradov's notation.

Definition 3.1. Lie algebra structures $\mathfrak{g}_1$ and $\mathfrak{g}_2$ on a vector space V are called compatible if $[,]_{\mathfrak{g}_1} + [,]_{\mathfrak{g}_2}$ is also a Lie algebra product.

A Lie algebra $\mathfrak{g}$ is called simply assembled from Lie algebra structures $\mathfrak{g}_1, \dots, \mathfrak{g}_m$ on $|\mathfrak{g}| = V$ if the Lie algebras $\mathfrak{g}_i$'s are pairwise compatible and $[,]_{\mathfrak{g}} = [,]_{\mathfrak{g}_1} + \dots [,]_{\mathfrak{g}_m}$. Note that if the Lie algebras $\mathfrak{g}_i$'s are compatible, then any linear combination of compatible Lie algebras commutators is a Lie algebra commutator (or product).

Definition 3.2. Fix a basis $B = e_1, \dots e_n$ in the representation vector space of a given Lie algebra. Let i, j, and k be integers, $1 \leq i, j, k \leq n$, no two of them equal, and denote by $\{i, j|k\}$ (respectively, $\{i|j\}$) the n-triadon (respectively, the n-dyon) such that $[e_i, e_j] = -[e_j, e_i] = e_k$ (respectively, $[e_i, e_j] = -[e_j, e_i] = e_j$) are the only non-trivial Lie commutators of basis vectors. Vinogradov called them 'base triadon' and 'base dyon', respectively or by the unifying term 'base lieon'.

An n-dyon is the direct sum of a dyon with an $n-2$-dimensional abelian Lie algebra, $n \geq 2$, (*i.e.* there is only one non vanishing bracket and it is a dyon). Analogously an n-triadon is the direct sum of a triadon with an $n-3$-dimensional abelian Lie algebra, $n \geq 3$ (*i.e.* there is only one non vanishing bracket and it is a triadon). They can also be referred generically as n-lieons.

A linear combination of pairwise compatible base lieons is called a *coaxial Lie algebra structure*. A Lie algebra structure will be called *trix-coaxial* (respectively, *dyx-coaxial*) if it consists only of base triadons (respectively, base dyons). A coaxial Lie algebra $\mathfrak{g}$ may be presented as a linear combination,

$$\mathfrak{g} = \sum \alpha_{(i,j|k)}\{i, j|k\} + \sum \beta_{(m|n)}\{m|n\}$$

of pairwise compatible base lieons.

The vectors e_i, e_j, and e_k (respectively, e_i, e_j) are called the *vertices of the triadon* $\{i, j|k\}$ (respectively, *of the dyon* $\{i|j\}$). The vectors e_i and e_j are called the *ends of the triadon* $\{i, j|k\}$, while e_k is the *center of the triadon*. The *origin and the end of the dyon* $\{i|j\}$ are e_i and e_j, respectively. The base triadons $\{i, j|k\}$ and $\{j, i|k\} = -\{i, j|k\}$ are not distinguished since they have identical compatibility properties.

We now recall Proposition 3.1 of [40] stating some necessary and sufficient conditions for the compatibility or incompatibility of particle-like Lie algebra structures:

- Two base triadons are non-trivially compatible if and only if they have a common center, a common end, or both.
- Two base dyons are incompatible if and only if the origin of one is the end of the other and they have no other common vertices.
- A base dyon is non-trivially compatible with a base triadon if and only if its origin coincides with one of the ends of the triadon.

For further notation and vocabulary we refer the reader to Vinogrados's papers.

3.1. Trix-coaxial, dyx-coaxial and particle-like Lie algebra structures for the Toda system

We prove that with a $(2+1)$-dimensional Toda type system are associated algebraic skeletons which are compatible assemblings of particle-like Lie algebras of dyons and triadons type. We obtain trix-coaxial and dyx-coaxial Lie algebra structures for the system from skeletons of some particular choice for compatible associated absolute parallelisms. In particular, we find a trix-coaxial Lie algebra structure made of two (compatible) base triadons constituting a 2-catena (see Proposition 3.1, pag 5 [40]).

Let us indeed now look for special skeletons.

3.2. Trix-coaxial Lie algebra structures

Proposition 3.3. *Associate with the Toda type system (1) is a trix-coaxial Lie algebra structure made of two (compatible) base triadons constituting a 2-catena.*

Proof. If we look for operators $P(u,\xi) = e^u \bar{P}(\xi)$, $M(u,\xi) = M(e^u,\xi)$, we get $M(e^u;\xi) = -e^u[L(\xi), \bar{P}(\xi)] + \bar{M}(\xi)$ and thus $\bar{P}(\xi) = -e^u[L(\xi),[L(\xi),\bar{P}(\xi)]] + [L(\xi),\bar{M}(\xi)]$. There are additional relations determined by the third prolongation equation $[-e^u[L(\xi),\bar{P}(\xi)] + \bar{M}(\xi), e^u\bar{P}(\xi)] = 0$.

Let us then put $L = X_1$, $\bar{M} = X_2$, $\bar{P} = X_3$, $[X_1, X_3] = X_4$. From the above we have the following prolongation closed Lie algebra

$$[X_1, X_2] = X_3\,, [X_1, X_3] = X_4\,, [X_1, X_4] = [X_2, X_3] = [X_2, X_4] = [X_3, X_4] = 0\,.$$

The above is a trix-coaxial Lie algebra structure made of two compatible 4-triadons.

Indeed, by taking $X_4 = 0$, we get $[X_1, X_2] = X_3\,, [X_1, X_3] = [X_2, X_3] = 0$ and $[X_1, X_4] = [X_2, X_4] = [X_3, X_4] = 0$ trivially.

On the other hand by taking $X_2 = 0$, we get $[X_1, X_3] = X_4\,, [X_1, X_4] = [X_3, X_4] = 0$ and $[X_1, X_2] = [X_2, X_3] = [X_2, X_4] = 0$ trivially.

According to [40] the two 4-triadons above *are non trivially compatible having a common end* X_1, and they constitute a 2-catena. □

3.2.1. Conservation laws and special solutions associated with a 2-catena

Let us now explicate the tower corresponding to such 4-triadons.

For the sake of simplicity let us put $A^k_m = B^k_m = \delta^k_m$, were δ^k_m is the Kronecker symbol. By substituting the above commutators into equations (8) and (10) (the expression of (9) being constrained in this case by the relation $F = G + H$), we get

$$H = e^u u_z X_1 + e^u X_3\,, \tag{12}$$

$$G = u_x X_1 - e^u [X_1, X_3] + X_2\,. \tag{13}$$

Now from equation (3), by sectioning we obtain

$$H^k - \xi^k_y = -\xi^k_x\,, \tag{14}$$

$$G^k + \xi^k_y = \xi^k_z\,, \tag{15}$$

(together with $F^k + \xi^k_x = \xi^k_z$ which depends on the two others).

We note that each one of 4-triadons above can be represented in a space of local coordinates ξ^k providing conservation laws related to two compatible Poisson structures.

Indeed let us consider the 4-triadon given by $X_2 = 0$. A representation in the coordinates $\{\xi^1, \xi^2, \xi^3\}$ is given by $X_1 = \xi^2\partial/\partial\xi^1$, $X_3 = \xi^3\partial/\partial\xi^2$ and $X_4 = -\xi^3\partial/\partial\xi^1$.

The tower corresponding to this case gives

$$e^u u_z \xi^2 \partial/\partial\xi^1 + e^u \xi^3 \partial/\partial\xi^2 - \xi^k_y \partial/\partial\xi^k = -\xi^k_x \partial/\partial\xi^k\,,$$
$$u_x \xi^2 \partial/\partial\xi^1 + e^u \xi^3 \partial/\partial\xi^1 + \xi^k_y \partial/\partial\xi^k = \xi^k_z \partial/\partial\xi^k\,,$$

which gives us the system

$$\boldsymbol{\xi}_x = \boldsymbol{\xi}_y + \boldsymbol{M}\boldsymbol{\xi} \tag{16}$$

$$\boldsymbol{\xi}_z = \boldsymbol{\xi}_y + \boldsymbol{N}\boldsymbol{\xi} \tag{17}$$

where $\boldsymbol{\xi} = (\xi^1, \xi^2, \xi^3)^T$, $\boldsymbol{M}$ and $\boldsymbol{N}$ are 3×3 matrices such that $M_{12} = -e^u u_z$, $M_{23} = -e^u$, $N_{12} = u_x$, $N_{13} = e^u$, and all the other entries are zeros. In view of (4)-(6) and (2) this system can be interpreted as a conservation law. It is a first order system which can be manipulated by resorting to the method of characteristics so that it turns out to be useful to find out special solutions of the Toda system.

Note indeed that this system is equivalent to the following system of coupled equations of Maxwell type

$$\xi^1_{yy} - \xi^1_{xx} = (e^u u_z \xi^2)_x + (e^u u_z \xi^2)_y \,, \tag{18}$$
$$\xi^2_{yy} - \xi^2_{xx} = (e^u \xi^3)_x + (e^u \xi^3)_y \,, \tag{19}$$
$$\xi^1_{zz} - \xi^1_{yy} = (e^u \xi^3 + u_x \xi^2)_z + (e^u \xi^3 + u_x \xi^2)_y \,, \tag{20}$$

where $e^u u_z \xi^2$, $e^u \xi^3$ and $e^u \xi^3 + u_x \xi^2$ can be recognized as charge/current densities. On the other hand, the system can be simplified since $\xi^2_y = \xi^2_z$; we have that (19) can also be written as

$$\xi^2_{zz} - \xi^2_{xx} = (e^u \xi^3)_x + (e^u \xi^3)_y \,, \tag{21}$$

from which we obtain the Maxwell-type equation

$$\xi^2_{yy} = \xi^2_{zz} \,. \tag{22}$$

Further manipulations can be made by using $\xi^3_y = \xi^3_y = \xi^3_z$.

We remark that the same conservation law and related outcomes are obtained by the 4-triadon given by $X_4 = 0$. Therefore existence of that tower with a finite dimensional skeleton which is a 2-catena says us that the two Poisson structures corresponding to each 4-triadon are compatible also in a sense which is interpretable from a physical point of view: they are structures associated with the same Toda system, and more precisely with the same conservation law and related special solutions. Compatibility of Poisson structures is beyond the scope of this paper, however this result suggests interesting links between special solutions and compatible Poisson structures, which will be the object of further investigations (in particular for meron-like configurations or gravitational instantons).

3.3. Dyx-coaxial and particle-like Lie algebra structures

In the following we analyze with more detail the case of choice $P(u, \xi) = \ln u \bar{P}(\xi)$, $M(u, \xi) = M(e^u, \xi)$ studied in [29] also leading to an infinite dimensional skeleton homomorphic to a Kač-Moody Lie algebra. We carefully distinguish the various cases.

This choice of the absolute parallelism associates with the Toda system (1) dyx-coaxial and particle-like Lie algebra structures.

First we need a preliminary result (see also [29]).

Lemma 3.4. *Let $P(u, \xi) = \ln u \bar{P}(\xi)$, $M(u, \xi) = M(e^u, \xi)$. We get the following infinitesimal algebraic skeleton with the structure of an* open Lie algebra*:*

$$\begin{aligned} &[X_1, X_2] = X_4 \,, [X_1, X_3] = X_5 \,, [X_4, X_5] = [X_2, X_7] \,, [X_3, X_4] = [X_2, X_5] \,, \\ &[X_1, X_4] = X_6 \,, [X_1, X_5] = X_7 \,, [X_2, X_3] = X_8 \,, \\ &[X_1, X_8] = [X_2, X_4] = [X_2, X_6] = [X_3, X_7] = 0 \,, \ldots \end{aligned} \tag{23}$$

Proof. Put $L = X_1(\xi)$.

By derivation we get $M(e^u, \xi) = -(\ln u - 1)u[X_1(\xi), X_3(\xi)] + X_2(\xi)$, and $P(u, \xi) = ue^u \ln u[X_1(\xi), M]$.

For $u \neq 0, 1$ (which are trivial solutions of the Toda system), from $[P, M] = 0$ we get

$$[[X_1, M], M] = 0\,;$$

from which we get

$$[[X_1, X_2], X_2] = 0\,,\ [X_1, [X_1, X_3]], X_2] + [[X_1, X_2], [X_1, X_3]] = 0\,,$$
$$[[X_1, [X_1, X_3]], [X_1, X_3]] = 0\,.$$

By putting $[X_1, X_2] = X_4$, $[X_1, X_3] = X_5$, $[X_1, X_4] = X_6$, $[X_1, X_5] = X_7$, $[X_2, X_3] = X_8$, we obtain an infinite dimensional skeleton as follows

$$\begin{aligned} &[X_1, X_8] = [X_2, X_4] = [X_2, X_6] = [X_3, X_7] = 0\,,\\ &[X_4, X_5] = [X_2, X_7]\,, [X_3, X_4] = [X_2, X_5]\,, \\ &\ldots \quad \square \end{aligned} \tag{24}$$

Here the dots means that we can continue this structure by introducing new generators still obtaining the peculiar relations of the type (24) which distinguish this algebraic structure from a freely generated Lie algebra (see the discussion in [26]).

Proposition 3.5. *The homomorphism $X_4 = \lambda X_2$ and $X_5 = \mu X_3$ associates with the Toda system (1) dyx-coaxial and particle-like Lie algebra structures as well as an infinite-dimensional Lie algebra homomorphic with a Kač-Moody Lie algebra.*

Proof. We essentially distinguish the two cases $X_8 \neq 0$ and $X_8 = 0$, together with various different subcases.

1. if $[X_2, X_3] = X_8 \neq 0$, then $\mu = -\lambda$ must old; we can distinguish different subcases
   (a) in general the case $X_8 \neq 0$ and $\mu = -\lambda \neq 0$ can provide infinite-dimensional Lie algebras homomorphic with Kač-Moody type Lie algebras.
   (b) the particular case $X_8 = \nu X_3$ and $\mu = -\lambda = 1$ (*i.e.* $X_4 = X_2$ and $X_5 = -X_3$) giving an infinite-dimensional Lie algebra homomorphic with a Kač-Moody Lie algebra was obtained in [29].
   (c) the particular case $X_8 = \nu X_3$ and $\mu = -\lambda = 0$ (*i.e.* $X_4 = X_5 = 0$; see [29]) gives a particle-like Lie algebra as a base 3-dyon:
   $$[X_1, X_2] = 0\,, [X_1, X_3] = 0\,, [X_2, X_3] = \nu X_3\,. \tag{25}$$
2. if $[X_2, X_3] = X_8 = 0$, then $X_6 = X_4$, $X_7 = X_5$, and we distinguish the following different subcases (the case $\mu = \lambda = 0$ giving an abelian Lie algebra):
   (a) the case $\mu = 0$ and $\lambda \neq 0$ provides us with a particle-like Lie algebra as a base 3-dyon:
   $$[X_1, X_2] = \lambda X_2\,, [X_1, X_3] = 0\,, [X_2, X_3] = 0\,. \tag{26}$$
   (b) the case $\lambda = \mu \neq 0$ provides a dyx-coaxial Lie algebra structure as an assembling of two compatible base 3-dyons
   $$[X_1, X_2] = \lambda X_2\,, [X_1, X_3] = \lambda X_3\,, [X_2, X_3] = 0\,. \tag{27}$$

(c) the particular case $\lambda = \mu = 1$ (*i.e.* $X_4 = X_2$ and $X_5 = X_3$) gives

$$[X_1, X_2] = X_2\,, [X_1, X_3] = X_3\,, [X_2, X_3] = 0\,,$$

and it was obtained in [29]. $\square$

Remark 3.6. We can try to check the compatibility of the above particle-like Lie algebra structures. The latter Lie algebra (27) is constituted of two mutually compatible dyons (dyx-coaxial Lie algebra), whose the first is given by (26), while the Lie algebras (25) and (26) are made of a single dyon and they are not compatible. Indeed we note that the first dyon of (27) is not compatible with (25), while the second dyon of (27) is.

The question now is can we still construct a different dyx-coaxial Lie algebra from the original skeleton? For example the following would be a dyx-coaxial Lie algebra of compatible dyons

$$[X_1, X_3] = \lambda X_3, [X_2, X_3] = \nu X_3, [X_1, X_2] = 0 \tag{28}$$

We ask if we can get it from the prolongation skeleton by a suitable quotienting, *i.e.* if it is somehow compatible with (or derivable from) the skeleton structure. However, we note that if we put $X_4 = 0$ from the beginning (which is the case when we assume that $[X_1, X_2] = 0$), and if $X_8 = \nu X_3$, this would imply also $[X_1, X_3] = 0$, then we would get (25) back.

Thus it appears that the case $X_8 = \nu X_3$ corresponds or to a particle-like Lie algebra structure or to a Kač-Moody type Lie algebra (it is noteworthy that the latter is anyway an infinite-dimensional loop Lie algebra of a dyx-coaxial Lie algebra) and these two cases appear to be non compatible.

Let us then investigate from a more general point of view this feature. We ask whether we can look for different quotient homomorphisms.

Let now consider the case $X_4 = 0$ from the beginning, and $X_8 \neq 0$, and look for a quotient lie algebra given by $X_8 = -\gamma X_2$, $X_5 = \mu X_3$, and we obtain the Lie algebra structure depending on two parameters

$$[X_1, X_2] = 0\,, [X_1, X_3] = \mu X_3\,, [X_3, X_2] = -\gamma X_2\,. \tag{29}$$

By applying the Jacobi identity we get $\mu\gamma X_2 = 0$ which, if we require $X_2 \neq 0$, is verified either for $\mu = 0$ and $\gamma \neq 0$ (see below (30)) or for $\mu \neq 0$ and $\gamma = 0$ (see below (31)), or for $\mu = 0$ and $\gamma = 0$ (trivial case of an abelian Lie algebra).

Proposition 3.7. *The case $X_4 = 0$ from the beginning, and with $X_8 \neq 0$, provides us with two base 3-dyons.*

Proof. 1. the case with $\mu = 0$ and $\gamma \neq 0$.
By putting $X_8 = -\gamma X_2$, and $X_4 = X_5 = X_6 = X_7 = 0$ we get the 3-dyon

$$[X_1, X_2] = 0\,, [X_1, X_3] = 0\,, [X_3, X_2] = -\gamma X_2\,. \tag{30}$$

The above dyon is incompatible with (25) while it is compatible with (26).

2. the case $\mu \neq 0$ and $\gamma = 0$.
We get the 3-dyon

$$[X_1, X_2] = 0\,, [X_1, X_3] = \mu X_3\,, [X_3, X_2] = 0\,. \quad \square \tag{31}$$

Remark 3.8. It appears that the dyx-coaxial Lie algebra (27) can be assembled by one step from the case (26) and the latter one, (31), by putting $\mu = \lambda$.

We note in particular that (29) can not be seen as a dyx-family of dyons since the two dyons $[X_1, X_3] = \mu X_3$, $[X_3, X_2] = -\gamma X_2$ are incompatible and indeed if we apply the Jacobi identity we get particle-like structures made of single base dyons.

It seems therefore that the prolongation skeleton is homomorphic with quotient finite dimensional Lie algebras which have always the structure of a family of compatible dyons or single base 3-dyons. We note that the first dyon of (27) is compatible with (30), while the second dyon of (27) is not.

Summing up we were able to associate with the infinitesimal skeleton (23) a dyx-coaxial Lie algebra structure (27) and particle-like Lie algebra structures made of three base 3-dyons which are only partially compatible among them, *i.e.*

- the first dyon of (27) is compatible with (30), while the second dyon of (27) is not.
- the first dyon of (27) is not compatible with (25), while the second dyon of (27) is.

Note that (25), (26), (30) and (31) are not all compatible among them, even they are not compatible in triples, but they are only compatible when took in couples.

3.3.1. A Lax pair associated with the dyx-coaxial Lie algebra structure of two compatible base 3-dyons

Following a procedure similar to that of Section 3.2.1 we shall now derive a Lax pair related to the Lie algebra (27) to which the tower skeleton (24) is homomorphic.

We refer again to equations (8) and (10). The tower associated with case 2.(*b*) in Proposition 3.5 becomes in this specific case

$$H = e^u u_z X_1 - u^2 e^u \ln u(\ln u - 1)[\, X_1, [X_1, X_3]] + u e^u \ln u [X_1, X_2]\,, \tag{32}$$

$$G = u_x X_1 - u(\ln u - 1)[X_1, X_3] + X_2\,. \tag{33}$$

By taking into account the representation ρ given by relations (27) we then get

$$H = e^u u_z X_1 - \lambda^2 u^2 e^u \ln u(\ln u - 1) X_3 + \lambda u e^u \ln u X_2\,, \tag{34}$$

$$G = u_x X_1 - \lambda u(\ln u - 1) X_3 + X_2\,. \tag{35}$$

Now, let us represent the dyx-coaxial Lie algebra above in a space of ‘*pseudopotentials*’ ξ^k by $X_1 = -\xi^1 \partial/\partial\xi^1 + \xi^2 \partial/\partial\xi^2$, $X_2 = -\lambda\xi^3 \partial/\partial\xi^1$, $X_3 = \lambda\xi^2 \partial/\partial\xi^3$.

Again by sectioning the tower, equations (14) and (15) provide the following

$$\begin{aligned}
&e^u(u_z\xi^1 + \lambda^2 u \ln u \xi^3)\partial/\partial\xi^1 - e^u u_z \xi^2 \partial/\partial\xi^2 + \\
&+\lambda^3 u^2 e^u \ln u(\ln u - 1)\xi^2 \partial/\partial\xi^3 + \xi^k_y \partial/\partial\xi^k = \xi^k_x \partial/\partial\xi^k\,, \\
&-u_x(\xi^1 + \lambda\xi^3)\partial/\partial\xi^1 + u_x \xi^2 \partial/\partial\xi^2 - \lambda^2 u(\ln u - 1)\xi^2 \partial/\partial\xi^3 + \\
&+\xi^k_y \partial/\partial\xi^k = \xi^k_z \partial/\partial\xi^k\,,
\end{aligned}$$

which gives us the inverse spectral problem

$$\boldsymbol{\xi}_x = \boldsymbol{\xi}_y + \hat{\boldsymbol{M}}\boldsymbol{\xi} \tag{36}$$

$$\boldsymbol{\xi}_z = \boldsymbol{\xi}_y + \hat{\boldsymbol{N}}\boldsymbol{\xi} \tag{37}$$

Where $\boldsymbol{\xi} = (\xi^1, \xi^2, \xi^3)^T$, $\hat{\boldsymbol{M}}$ and $\hat{\boldsymbol{N}}$ are 3×3 matrices such that $\hat{M}_{11} = e^u u_z$, $\hat{M}_{13} = e^u \lambda^2 u \ln u$, $\hat{M}_{22} = -e^u u_z$, $\hat{M}_{32} = \lambda^3 u^2 e^u \ln u(\ln u - 1)$ $\hat{N}_{11} = -u_x$, $\hat{N}_{13} = -\lambda u_x$, $\hat{N}_{22} = u_x$, $\hat{N}_{32} = -\lambda^2 u(\ln u - 1)$ and all the other entries are zeros. Here λ plays the role of a spectral parameter and, in view of (4)-(6) and (2), $\hat{\boldsymbol{M}}$ and $\hat{\boldsymbol{N}}$ can be considered a Lax pair related to the multidimensional Toda system (1) (for spectral problems related to multidimensional nonlinear system see, *e.g.* [22,38]). This Lax pair should be compared with [19].

Compatibility of Lie algebraic structures being expressions of compatibility of the corresponding Poisson structures, we note here that Fernandes [6] studied the relationship between the master symmetries and bi-Hamiltonian structure of the Toda lattice. We stress that *dyons provide indeed particular examples of master symmetries* of related ordinary differential equations.

3.4. Concluding remarks

The structure of trix-coaxial and dyx-coaxial Lie algebras assembled in one step from couples of particle-like Lie algebra structures appears as an intrinsic feature of the Toda system (1), at least associated with the chosen absolute parallelisms. Indeed the similitude transformations seems to be the fundamental internal symmetries of the system (see *e.g.* [2]).

As final remark, since (30) is compatible with (26), and since (25) is compatible with (31), we could construct the following dyx-coaxial Lie algebras:

$$[X_1, X_2] = \lambda X_2 \, , [X_1, X_3] = 0 \, , [X_3, X_2] = -\gamma X_2 \, , \tag{38}$$

and

$$[X_1, X_2] = 0 \, , [X_1, X_3] = \mu X_3 \, , [X_2, X_3] = \nu X_3 \, . \tag{39}$$

However, it is important to realize that they could *not* be obtained from the skeleton (23) by the choice of an homomorphism, and therefore they are not identified as internal symmetries of the Toda system by the choice of the absolute parallelism given by Lemma 3.4. The question if the choice of other forms of the absolute parallelism could identify them is open and will be the object of future investigations.

CRediT authorship contribution statement

The Authors contribution is equal in all parts of the paper.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

Research partially supported by Department of Mathematics - University of Torino through the projects PALM_RILO_16_01 and FERM_RILO_17_01 (MP) and written under the auspices of GNSAGA-INdAM. The first author (MP) would like to acknowledge the contribution of the COST Action CA17139. The authors would like to thank M.V. Pavlov for bringing to their attention the reference [45].

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