Geometric integration by parts and Lepage equivalents*

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Abstract

We compare the integration by parts of contact forms - leading to the definition of the interior Euler operator - with the so-called canonical splittings of variational morphisms. In particular, we discuss the possibility of a generalization of the first method to contact forms of lower degree. We define a suitable Residual operator for this case and, working out an original idea conjectured by Olga Rossi, we recover the Krupka-Betounes equivalent for first order field theories. A generalization to the second order case is discussed.

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1 Introduction

The Euler–Lagrange operator can be geometrically described by means of two interrelated geometric objects (and corresponding geometric integration

*Our colleague and friend Olga Rossi passed away in October 2019. This paper is an outcome of our collaboration, which we miss heartily, and we dedicate it to her memory.

by parts procedures), the one based on the concept of differential forms and exterior differential modulo contact structures, the other based on the interpretation of variational objects as fibered morphisms [7, 8, 3] etc.

Following an approach inaugurated by the works of Cartan and Lepage, the finite order variational sequence was introduced and developed by Krupka; see *e.g.* [12], [13] and [14]. The problem of the representation of the finite order variational sequence (whose objects, we recall, are equivalence classes of local differential forms) has been discussed in terms of the so called *interior Euler operator*; see *e.g.* [9] and [18].

On the other hand variational morphisms [3] not only provide a geometric formulation of the calculus of variations, but in general of a wide class of differential operators. Their most relevant property is that they admit canonical and algorithmic splittings and, by the introduction of a connection on the base manifold and a connection on the considered fiber bundle, globality and uniqueness properties of these splittings can be assured.

The aim of this paper is to investigate the relation between these two approaches which use different geometric integration by parts techniques.

We perform the identification of contact forms with variational morphisms for 1-contact forms of degree at most $n+1$, where n is the dimension of the base manifold of the considered fiber bundles. The two integration by parts techniques are directly compared in the case of 1-contact n-horizontal $(n+1)$-forms. A non-trivial manipulation of the interior Euler operator technique is needed for the comparison in the case of 1-contact forms of lower degree.

We will show that in the first situation the two approaches produce the same results, whilst in the second situation we get two different splittings and we can give an account of their difference. Finally we observe that the manipulations needed in this last case might be useful in order to obtain an extension of the definition of the Krbek–Musilová interior Euler operator at least to 1-contact forms of lower degree. We define a suitable Residual operator for this case and we recover the Krupka-Betounes Lepage equivalent for first order field theories.

2 Contact structure and geometric integration by parts

Prolongations of fibered manifolds are a basic tool for the geometric formulation of the calculus of variations. We recall the decomposition of pull-backs of differential forms on jet prolongations of fibered manifolds. The decomposition is performed by means of the jet projections and the holonomic lift of tangent vectors (a canonical construction which allows a splitting of the projection along the affine fibrations defining the contact structure of tangent vectors in two components with remarkable properties). Moreover the decomposition of forms leads to the introduction of the so-called contact forms, which reveal to be another fundamental concept in the calculus of variations. To fix notation we follow [15] ; other references on jet spaces are [22] and [8].

We recall that by a *fibered manifold structure* on a C^∞ manifold Y we mean a triplet (Y, X, π), where X is a C^∞ manifold called the *base* and $\pi\colon Y \longrightarrow X$ is a surjective submersion of class C^∞ called the *projection*. We stress that, when dealing with local aspects of fibered manifolds, we will always use the so-called *fibered charts* (i.e. charts adapted to the fibration). Let Y be a fibered manifold with base X and projection π, let $n = \mathrm{dim}X$ and $m = \mathrm{dim}Y - n$. We denote by J^rY, where $r \geq 0$ is any integer, the set of r-jets $J^r_x\gamma$ of C^r *sections* of Y with source $x \in X$ and target $y = \gamma(x) \in Y$ (for more details on jet spaces see [22] and [8]); we fix the notation $J^0Y = Y$. For any s such that $0 \leq s \leq r$ we have surjective mappings, the *canonical jet projections*, $\pi^r_s\colon J^rY \longrightarrow J^sY$ and $\pi^r\colon J^rY \longrightarrow X$, defined by $\pi^r_s(J^r_x\gamma) = J^s_x\gamma$, $\pi^r(J^r_x\gamma) = x$. Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y and let (U, φ), $\varphi = (x^i)$, be the associated chart on X.

By setting $V^r = (\pi^r_0)^{-1}(V)$, a chart on the set J^rY *associated* with the fibered chart (V, ψ) is given by (V^r, ψ^r) $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \dots, y^\sigma_{j_1j_2\dots j_r})$, with $1 \leq i \leq n$, $\quad 1 \leq \sigma \leq m$, $1 \leq j_1 \leq j_2 \leq \dots \leq j_k \leq n$, $k = 1, 2, 3, \dots, r$. The set of associated charts (V^r, ψ^r), such that the fibered charts (V, ψ) constitute a smooth atlas on Y, is a smooth atlas on J^rY. With this smooth structure J^rY is called the *r-jet prolongation* of the fibered manifold Y.

Let Y be a fibered manifold with base X and projection π. Let Ξ be a π-projectable vector field on Y, expressed in a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, by $Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma}$, then its s-th prolongation $J^s\Xi$ is expressed in the associated chart (V^s, ψ^s) by

$$J^s\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma} + \sum_{k=1}^{s} \sum_{j_1 \le j_2 \le \dots \le j_k} \Xi^\sigma_{j_1 j_2 \dots j_k} \frac{\partial}{\partial y^\sigma_{j_1 j_2 \dots j_k}},$$

where $\Xi^\sigma_{j_1 j_2 \dots j_k} = d_{j_k} \Xi^\sigma_{j_1 j_2 \dots j_{k-1}} - y^\sigma_{j_1 j_2 \dots j_{k-1} i} \frac{\partial \xi^i}{\partial x^{j_k}}$.

Let $J^{r+1}_x\gamma \in J^{r+1}Y$. To any tangent vector ξ of $J^{r+1}Y$ at the point $J^{r+1}_x\gamma$ is assigned a tangent vector of J^rY at the point $\pi^{r+1}_r(J^{r+1}_x\gamma) = J^r_x\gamma$ by $h\xi := TJ^r\gamma \circ T\pi^{r+1}(\xi)$. We get a vector bundle morphism $h\colon TJ^{r+1}Y \longrightarrow TJ^rY$ over the jet projection π^{r+1}_r called the *horizontalization*, and $h\xi$ is called the *horizontal component* of ξ. Let ξ be given in a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$ as

$$\xi = \xi^i \frac{\partial}{\partial x^i}\bigg|_{J^{r+1}_x\gamma} + \sum_{k=0}^{r+1} \sum_{j_1 \le j_2 \le \dots \le j_k} \Xi^\sigma_{j_1 j_2 \dots j_k} \frac{\partial}{\partial y^\sigma_{j_1 j_2 \dots j_k}}\bigg|_{J^{r+1}_x\gamma},$$

then

$$h\xi = \xi^i d_i\,,$$

where $d_i = \frac{\partial}{\partial x^i}\Big|_{J^r_x\gamma} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \dots \le j_k} y^\sigma_{j_1 j_2 \dots j_k i} \frac{\partial}{\partial y^\sigma_{j_1 j_2 \dots j_k}}\Big|_{J^r_x\gamma}$, the i-th *formal derivative operator*, is a vector field along the projection π^{r+1}_r.

We can assign to every tangent vector $\xi \in T_{J^{r+1}_x\gamma}J^{r+1}Y$ a tangent vector $p\xi \in T_{J^r_x\gamma}J^rY$ by the decomposition $T\pi^{r+1}_r(\xi) = h\xi + p\xi$, where $p\xi$ is called the *contact component* of the vector ξ. Then

$$p\xi = \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \dots \le j_k} \left(\Xi^\sigma_{j_1 j_2 \dots j_k} - y^\sigma_{j_1 j_2 \dots j_k i}\xi^i\right) \frac{\partial}{\partial y^\sigma_{j_1 j_2 \dots j_k}}\bigg|_{J^r_x\gamma}.$$

For any open set $W \subset Y$, $\Omega^r_q W$ denotes the C^∞-module of q-forms on the open set $W^r = (\pi^r_0)^{-1}(W)$ in J^rY, and $\Omega^r W$ is the exterior algebra of differential forms on W^r. In order to study the structure of the components of a form $\rho \in \Omega^r_q W$, it will be convenient to introduce a *multi-index notation.* A *multi-index* I is an ordered k-tuple $I = (i_1 i_2 \dots i_k)$, where $k = 1, 2, \dots, r$ and the entries are indices such that $1 \le i_1, i_2, \dots, i_k \le n$. The number k is the *lenght* of I and is denoted by $|I|$. If $1 \le j \le n$ is any integer, we denote by Ij the multi-index $Ij = (i_1 i_2 \dots i_k j)$.

The notion of horizontalization of vectors can be used to define a morphism $h\colon \Omega^r W \longrightarrow \Omega^{r+1} W$ of exterior algebras.
Let $\rho \in \Omega^r_q W$, with $q \geq 1$, and $J^{r+1}_x\gamma \in W^{r+1}$. Consider the pullback $(\pi^{r+1}_r)^*\rho$ and the value

$$\begin{aligned}
&(\pi^{r+1}_r)^*\rho(J^{r+1}_x\gamma)(\xi_1, \xi_2, \dots, \xi_q) = \\
&= \rho(J^r_x\gamma)(T\pi^{r+1}_r(\xi_1), T\pi^{r+1}_r(\xi_2), \dots, T\pi^{r+1}_r(\xi_q))
\end{aligned} \tag{1}$$

on any tangent vectors $\xi_1, \xi_2, \dots, \xi_q$ of $J^{r+1}Y$ at the point $J^{r+1}_x\gamma$. Decompose each of these vectors into the horizontal and contact components, $T\pi^{r+1}_r(\xi_l) = h\xi_l + p\xi_l$, and set

$$h\rho(J^{r+1}_x\gamma)(\xi_1, \xi_2, \dots, \xi_q) := \rho(J^r_x\gamma)(h\xi_1, h\xi_2, \dots, h\xi_q).$$

This formula defines a q-form $h\rho \in \Omega^{r+1}_q W$, while for 0-forms $hf := (\pi^{r+1}_r)^* f$. It follows that $h\rho(J^{r+1}_x\gamma)(\xi_1, \xi_2, \dots, \xi_q)$ vanishes whenever at least one of the vectors is π^{r+1}-vertical. Thus, the q-form $h\rho$ must be π^{r+1}-*horizontal*. In particular $h\rho = 0$ whenever $q \geq n+1$. The component $h\rho$ is called the *horizontal component* of ρ. We say that $\rho \in \Omega^r_1 W$ is *contact* if $h\rho = 0$. Let us now set

$$\begin{aligned}
&p_k\rho(J^{r+1}_x\gamma)(\xi_1, \xi_2, \dots, \xi_q) := \\
&:= \frac{1}{k!(q-k)!} \sum_{\sigma \in \mathcal{P}_q} (-1)^{|\sigma|} \rho(J^r_x\gamma)(p\xi_{\sigma(1)}, \dots, p\xi_{\sigma(k)}, h\xi_{\sigma(k+1)}, \dots, h\xi_{\sigma(q)})
\end{aligned}$$

where $\mathcal{P}_q$ is the set of permutations of q elements and $|\sigma|$ is the sign of the permutation $\sigma \in \mathcal{P}_q$. Note that if $k = 0$, then $p_0\rho$ coincides with $h\rho$, while for 0-forms $pf = 0$. In particular, given a q-form η

$$\eta = \sum_{s=0}^{q} A^{J_1}_{\sigma_1} \dots {}^{J_s}_{\sigma_s i_{s+1} \dots i_q} \, dy^{\sigma_1}_{J_1} \wedge \dots \wedge dy^{\sigma_s}_{J_s} \wedge dx^{i_{s+1}} \wedge \dots \wedge dx^{i_q} \,,$$

the k-contact component of η has the chart expression

$$p_k\eta = B^{J_1}_{\sigma_1} \dots {}^{J_k}_{\sigma_k i_{k+1} \dots i_q} \, \omega^{\sigma_1}_{J_1} \wedge \dots \wedge \omega^{\sigma_k}_{J_k} \wedge dx^{i_{k+1}} \wedge \dots \wedge dx^{i_q},$$

with

$$B^{J_1}_{\sigma_1} \dots {}^{J_k}_{\sigma_k i_{k+1} \dots i_q} = \sum_{s=k}^{q} \binom{s}{k} A^{J_1}_{\sigma_1} \dots {}^{J_k J_{k+1}}_{\sigma_k \sigma_{k+1}} \dots {}^{J_s}_{\sigma_s [i_{s+1} \dots i_q} \, y^{\sigma_{k+1}}_{J_{k+1} i_{k+1}} \dots y^{\sigma_s}_{J_s i_s]}$$

where the antisymmetrization in the right hand side of the last equation is performed only on the indices $i_{k+1}\dots i_s i_{s+1}\dots i_q$.

For any $\rho\in\Omega^r_q W$, $q\geq 0$, the *canonical decomposition* of the form ρ is given as

$$(\pi^{r+1}_r)^*\rho = h\rho + p_1\rho + p_2\rho + \cdots + p_q\rho.$$

We can see that the canonical decomposition of forms gives rise to the splitting of the pull-back of the exterior derivative

$$(\pi^{r+2}_r)^* d\rho = d_H\rho + d_C\rho = d_H\rho := \sum_{k=0}^{q} p_k d p_k\rho + \sum_{k=0}^{q} p_{k+1} d p_k \rho\,,,$$

and characterized by the identities $d_H\circ d_H = 0$, $d_C\circ d_C = 0$, $d_C\circ d_H = -d_H\circ d_C$; furthermore, if ρ is a q-form and η is an s-form, both on J^rY, then

$$\begin{aligned} d_H(\rho\wedge\eta) &= d_H\rho\wedge(\pi^{r+2}_r)^*\eta + (-1)^q(\pi^{r+2}_r)^*\rho\wedge d_H\eta \\ d_C(\rho\wedge\eta) &= d_C\rho\wedge(\pi^{r+2}_r)^*\eta + (-1)^q(\pi^{r+2}_r)^*\rho\wedge d_C\eta. \end{aligned}$$

2.1 The interior Euler operator

We recall some technical features of the interior Euler operator, seen as a tool which allows to pass in a univocal way from equivalence classes of local differential forms in the variational sequence, to (global) differential forms in the representation sequence. First we recall the *finite order* variational sequence as introduced by Krupka in [12]. A complete description of this subject involves some topics of sheaf theory and sheaf cohomology; however, since our purpose is to make direct calculations on the representation of the variational sequence, we just refer to [15] for more details about those aspects. Then we shortly recall the notion of *Lie derivative* of forms and some results on integration by parts formulae which lead directly to the definition of the interior Euler operator. For more details and other related topics we refer to [9] and [18].

Let Ω^r_q, $q\geq 0$, be the direct image of the sheaf of smooth q-forms over J^rY by the jet projection π^r_0. We denote by

$$\Omega^r_{q,\mathrm{c}} = \begin{cases} \ker p_0 & \text{for } 1\leq q\leq n, \\ \ker p_{q-n} & \text{for } n+1\leq q\leq \dim J^rY \end{cases}$$

the sheaf of contact q-forms, if $q \leq n$, or the sheaf of strongly contact q-forms, if $n+1 \leq q \leq \dim J^rY$.

We set

$$\Theta^r_q = \Omega^r_{q,\mathrm{c}} + d\Omega^r_{q-1,\mathrm{c}}$$

where $d\Omega^r_{q-1,\mathrm{c}}$ is the image sheaf of $\Omega^r_{q-1,\mathrm{c}}$ by the exterior derivative d. Let us consider the sequence of sheaves

$$\{0\} \to \Theta^r_1 \to \cdots \to \Theta^r_n \to \Theta^r_{n+1} \to \cdots \to \Theta^r_P \to \{0\} \tag{2}$$

in which the arrows are given by exterior derivatives d and with P being the maximal nontrivial degree. It can be shown that it is an exact subsequence of the de Rham sequence. The resulting quotient sequence

$$\{0\} \to \mathbb{R}_Y \to \Omega^r_0 \to \Omega^r_1/\Theta^r_1 \to \Omega^r_2/\Theta^r_2 \to \Omega^r_3/\Theta^r_3 \to \dots$$

is called the *variational sequence of order* r and it is an acyclic resolution of the constant sheaf $\mathbb{R}_Y$ over Y. We denote the quotient mappings as follows

$$E^r_q \colon [\rho] \in \Omega^r_q/\Theta^r_q \longrightarrow E^r_q([\rho]) = [d\rho] \in \Omega^r_{q+1}/\Theta^r_{q+1}.$$

Note that, in particular, the mappings E^r_n and E^r_{n+1} correspond to the Euler-Lagrange mapping and to the Helmholtz-Sonin mapping of calculus of variations, respectively.

Definition 2.1. *Let (V,ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y and let ρ be a differential q-form on J^rY. The Lie derivative of a q-form ρ on with respect to a vector field $h\Xi$ along the map π^{r+1}_r is given by*

$$\pounds^{\pi^{r+1}_r}_{h\Xi}\rho = (\pi^{r+1}_r)^*_{h\Xi}(h\Xi \rfloor d\rho) + d(\pi^{r+1}_r)^*_{h\Xi}(h\Xi \rfloor \rho)\,.$$

Here $(\pi^{r+1}_r)^*_{h\Xi}$ is a pull-back defined according to [9].

In particular, let d_i be the i-th *formal derivative operator* seen as a (horizontal) vector field along a map. We have $\pounds^{\pi^{r+1}_r}_{\mathrm{d}_i} dx^j = 0$, $\pounds^{\pi^{r+1}_r}_{\mathrm{d}_i} dy^\sigma_J = dy^\sigma_{Ji}$, $\pounds^{\pi^{r+1}_r}_{\mathrm{d}_i} \omega^\sigma_J = \omega^\sigma_{Ji}$, while for f a zero form, we have $\pounds^{\pi^{r+1}_r}_{\mathrm{d}_i} f = \frac{\partial f}{\partial x^i} + \sum_{|J|=0}^r y^\sigma_{Ji}\frac{\partial f}{\partial y^\sigma_J} = d_i f$.

Accordingly, by a slight abuse of notation, we will use at any degree the symbol $\mathrm{d}_i = \pounds^{\pi^{r+1}_r}_{\mathrm{d}_i}$, and we will call it the *total derivative of forms* with respect to the coordinate x^i.

We recall that the total derivatives of forms enjoy the following properties

1. the form $\mathrm{d}_H\rho$ can be locally decomposed as
$$\mathrm{d}_H\rho = (-1)^q \mathrm{d}_i\rho \wedge dx^i$$
2. the Leibniz rule holds for total derivatives of the exterior product of forms ρ and η
$$\mathrm{d}_i(\rho \wedge \eta) = \mathrm{d}_i\rho \wedge \eta + \rho \wedge \mathrm{d}_i\eta$$
3. let $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{x}^j, \bar{y}^\nu)$, be a fibered chart on Y, such that $V \cap \bar{V} \neq \emptyset$ and let $\bar{\mathrm{d}}_j$ be the total derivative with respect to the coordinate $\bar{x}^j$. Then the transformation rule $\mathrm{d}_i\rho = \frac{\partial \bar{x}^j}{\partial x^i}\bar{\mathrm{d}}_j\rho$ holds.
4. the total derivatives commute, *i.e.*
$$\mathrm{d}_i\mathrm{d}_j\rho = \mathrm{d}_j\mathrm{d}_i\rho.$$

This last property allows us to use the notation $\mathrm{d}_J = \mathrm{d}_{j_s} \circ \cdots \circ \mathrm{d}_{j_1}$, where $J = (j_1 \dots j_s)$ is a multi-index.

In the following we consider a generalization of the integration by parts procedure to differential forms based on the concept of total derivative of forms [9].

Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y and $\rho \in \Omega^r_{n+k}V$ a form. Let $\mathrm{p}_k\rho$ be expressed as
$$\mathrm{p}_k\rho = \sum_{|J|=0}^{r} \omega^\sigma_J \wedge \eta^J_\sigma.$$
Then there exists the decomposition
$$p_k\rho = \mathcal{I}(\rho) + p_k dp_k \mathcal{R}(\rho) \tag{3}$$
where $\mathcal{I}$ is the *interior Euler operator*, $\mathcal{R}$ is the *Residual operator*, and $\mathcal{R}(\rho)$ is a local k-contact $(n+k-1)$-form.

There exists a unique decomposition as above such that $\mathcal{I}$ is $\mathbb{R}$-linear, which is therefore globally defined. In local coordinates we have
$$\mathcal{I}\colon \Omega^r_{n+k}W \ni \rho \longrightarrow \mathcal{I}(\rho) = \frac{1}{k}\omega^\sigma \wedge \sum_{|J|=0}^{r}(-1)^{|J|}\mathrm{d}_J(\frac{\partial}{\partial y^\sigma_J} \lrcorner\, p_k\rho) \in \Omega^{2r+1}_{n+k}W\,. \tag{4}$$

Let $W \subset Y$ be an open set and let $\rho \in \Omega^r_{n+k}W$, $1 \leq k \leq \dim J^rY - n$, be a form. Then the following intrinsic properties uniquely characte the interior Euler operator:

(a) $(\pi^{2r+1}_r)^*\rho - \mathcal{I}(\rho) \in \Theta^{2r+1}_{n+k}W$;

(b) $\mathcal{I}(p_k dp_k \mathcal{R}(\rho)) = 0$;

(c) $\mathcal{I}^2(\rho) = (\pi^{4r+3}_{2r+1})^*\mathcal{I}(\rho)$;

(d) $\ker(\mathcal{I}) = \Theta^r_{n+k}W$.

2.2 Variational morphisms and canonical splittings

We recall shortly the definition and the basic properties of variational morphisms; see [3]. Then, in view of a comparison with the Krbek-Musilová geometric integration by parts, we discuss in greater detail their algorithmic splitting properties, which correspond to the possibility of performing a global and *covariant* integration by parts, distinguishing the case of codegree $s = 0$ from the case $0 < s \leq n$. Finally we include some results about the uniqueness properties of the aforementioned splittings.

The general situation that we will take into account dealing with variational morphisms is described by the following:

Definition 2.2. *Let $\mathcal{E} = (E, X, \tilde{\pi}, \mathbb{R}^l)$ be a vector bundle and $\pi\colon Y \longrightarrow X$ an arbitrary fiber bundle, both over X, with dim$X = n$. Let t, r and s be integers. A bundle morphism*

$$\mathbb{V}\colon J^tY \longrightarrow (J^r\mathcal{E})^* \otimes A_{n-s}(X)$$

is called a variational $\mathcal{E}$-morphism on Y. The (minimal) integer t is called the order of $\mathbb{V}$, r is called the rank and $(n-s)$ is called the degree of $\mathbb{V}$ (being s the codegree).

A fibered connection on $\mathcal{E}$ (*i.e.* a linear connection Γ^a_{bi} on X and a connection Γ^A_{Bi} on $\mathcal{E}$) induces on $(J^r\mathcal{E})^* \otimes A_{n-s}(X)$ a set of local fibered coordinates $(x^i; \hat{v}^{i_1\dots i_s}_A, \dots, \hat{v}^{i_1\dots i_s j_1\dots j_r}_A)$ so that a variational morphism $\mathbb{V}$ can be locally given there as

$$< \mathbb{V}|J^r\Xi >= \frac{1}{s!}\Big[\hat{v}^{i_1\dots i_s}_A \hat{\Xi}^A + \hat{v}^{i_1\dots i_s j}_A \hat{\Xi}^A_j + \dots + \hat{v}^{i_1\dots i_s j_1\dots j_r}_A \hat{\Xi}^A_{j_1\dots j_r}\Big] \otimes ds_{i_1\dots i_s}$$

where $ds_{i_1\dots i_s} = \frac{\partial}{\partial x^{i_s}} \rfloor \dots \rfloor \frac{\partial}{\partial x^{i_1}} \rfloor ds$, if ds is the volume density on the base manifold X. Each coefficient $\mathbb{V}_m = \frac{1}{s!}\hat{v}^{i_1\dots i_s j_1\dots j_m}_A$ of order $0 \leq m \leq r$ is the coefficient of a global variational morphism, the *m-rank term of* $\mathbb{V}$ (if $m = r$ it is called the *highest rank term of* $\mathbb{V}$).

Let now $\mathbb{Q}\colon J^tY \longrightarrow A_{n-s}(X)$ be a morphism of rank $r = 0$. The *divergence* of $\mathbb{Q}$ is the variational morphism $\mathrm{Div}(\mathbb{Q})\colon J^{t+1}Y \longrightarrow A_{n-s+1}(M)$ such that

$$\mathrm{Div}(\mathbb{Q}) \circ J^{t+1}\sigma = d(\mathbb{Q} \circ J^t\sigma)$$

for each section $\sigma\colon X \longrightarrow Y$.

Variational morphisms admit canonical and algorithmic splittings corresponding to global and covariant integration by parts. We distinguish two cases.

- the case of codegree $s = 0$.

  Let $\mathbb{V}\colon J^tY \longrightarrow (J^r\mathcal{E})^* \otimes A_n(X)$ be a variational $\mathcal{E}$-morphism of codegree $s = 0$. Then we can define two global variational $\mathcal{E}$-morphisms

  $$\begin{aligned}\mathbb{E} \equiv \mathbb{E}(\mathbb{V})\colon J^{t+r}C &\longrightarrow \mathcal{E}^* \otimes A_n(X)\\ \mathbb{T} \equiv \mathbb{T}(\mathbb{V})\colon J^{t+r-1}C &\longrightarrow (J^{r-1}\mathcal{E})^* \otimes A_{n-1}(X)\end{aligned}$$

  such that the following splitting property holds true:

  $$< \mathbb{V}|J^r\Xi >=< \mathbb{E}|\Xi > +\mathrm{Div}(< \mathbb{T}|J^{r-1}\Xi >) \tag{5}$$

  for each section Ξ of $\mathcal{E}$. The variational morphism $\mathbb{E}$ is called the volume part of $\mathbb{V}$ while $\mathbb{T}$ is called the boundary part of $\mathbb{V}$.

In particular, we locally have:

$$< \mathbb{E}|\Xi >= \left[\left(\hat{v}_A - \nabla_{j_1}\hat{v}_A^{j_1} + \cdots + (-1)^r\nabla_{j_1\ldots j_r}\hat{v}_A^{j_1\ldots j_r}\right)\Xi^A\right] \otimes ds$$

and

$$< \mathbb{T}|J^{r-1}\Xi >= \left[\hat{t}_A^i\hat{\Xi}^A + \hat{t}_A^{ij_1}\hat{\Xi}_{j_1}^A + \cdots + \hat{t}_A^{ij_1\ldots j_{r-1}}\hat{\Xi}_{j_1\ldots j_{r-1}}^A\right] \otimes ds_i$$

where the coefficients of $\mathbb{T}$ are given by the recurrence relations

$$\begin{aligned}&\hat{t}_A^{ij_1\ldots j_{r-1}} = \hat{v}_A^{ij_1\ldots j_{r-1}}\\ &\hat{t}_A^{ij_1\ldots j_{r-2}} = \hat{v}_A^{ij_1\ldots j_{r-2}} - \nabla_l\hat{t}_A^{lij_1\ldots j_{r-2}}\\ &\ldots\\ &\hat{t}_A^i = \hat{v}_A^i - \nabla_l\hat{t}_A^{li}\,.\end{aligned} \tag{6}$$

A similar splitting formula can be obtained for variational morphisms of higher codegree. Let us first define the concept of a reduced morphism with respect to a fibered connection.

Definition 2.3. *Let* $\mathbb{V}\colon J^tY \longrightarrow (J^r\mathcal{E})^*\otimes A_{n-s}(X)$ *be a variational morphism.*
Let $\mathbb{V}_m = \frac{1}{s!}(\hat{v}_A^{i_1\dots i_s j_1\dots j_m})\otimes ds_{i_1\dots i_s}$ *be the coefficient of its term of rank* $0\le m\le r$.

The term $\mathbb{V}_m$ *is said to be reduced with respect to the fibered connection* $(\Gamma^a_{bi},\Gamma^A_{Bi})$ *if* $\hat{v}_A^{[i_1\dots i_s j_1]j_2\dots j_m}=0$. *The variational morphism* $\mathbb{V}$ *is reduced if all its terms are reduced.*

Notice that when $n=\dim(X)=1$, *e.g.* in the case of Mechanics, all variational morphisms are reduced. However, a variational morphism might be reduced with respect to a fibered connection, but not with respect to another fibered connection, whenever its rank is at least two.

- the case of codegree $s\ge 1$.

  Let now $\mathbb{V}\colon J^tY \longrightarrow (J^r\mathcal{E})^*\otimes A_{n-s}(X)$ be a global variational $\mathcal{E}$-morphism of codegree $s\ge 1$. Then we can define two global variational $\mathcal{E}$-morphisms

  $$\begin{aligned}\mathbb{E}\equiv\mathbb{E}(\mathbb{V})&\colon J^{t+r}C\longrightarrow (J^r\mathcal{E})^*\otimes A_{n-s}(X)\\ \mathbb{T}\equiv\mathbb{T}(\mathbb{V})&\colon J^{t+r-1}C\longrightarrow (J^{r-1}\mathcal{E})^*\otimes A_{n-s-1}(X)\end{aligned}$$

  where $\mathbb{E}$ is a reduced variational morphism and such that the following holds true:

  $$<\mathbb{V}|J^r\Xi>=<\mathbb{E}|J^r\Xi>+\mathrm{Div}(<\mathbb{T}|J^{r-1}\Xi>) \tag{7}$$

  for each section Ξ of $\mathcal{E}$. Again, the variational morphism $\mathbb{E}$ is called the volume part of $\mathbb{V}$ while $\mathbb{T}$ is called the boundary part of $\mathbb{V}$.

Let a fibered connection be fixed; the volume part is uniquely determined, while the boundary part is determined modulo a divergenceless term. When $r\ge 2$ one can proceed by further splitting thus obtaining

$$<\mathbb{T}|J^{r-1}\Xi>=<\mathbb{S}|J^{r-1}\Xi>+\mathrm{Div}(<\mathbb{Q}|J^{r-2}\Xi>)\,, \tag{8}$$

where the variational morphism $\mathbb{S}\colon J^{t+2r-2}Y\longrightarrow (J^{r-1}\mathcal{E})^*\otimes A_{n-s-1}(X)$ is reduced by construction. Although by two different splittings of the variational morphism $\mathbb{V}$ the volume part is uniquely determined while the boundary parts are not, the reduced part of the boundary parts is uniquely determined as well; one then speaks of a "canonical" boundary term in this sense.

3 Comparison of the two approaches and new results

We present some original results which clarify the similarities and differences between the two integration by parts methods described above.

The basic idea is that 1-contact forms of degree $n+1$ can be seen as variational morphisms and *viceversa* to each variational morphism a 1-contact form of degree $n+1$ can be associated. In Proposition 3.1, we prove the equivalence of decompositions (3) and (5) for 1-contact $(n+1)$-forms (which we shall call *top forms*, because they are of the highest horizontal degree), seen as variational morphisms of codegree $s = 0$.

Then the more difficult case of 1-contact $(n-s+1)$-forms, seen as variational morphisms of codegree $0 < s \leq n$, is discussed. We show that, in general, for k-contact $(n-s)$-horizontal $(n-s+k)$-forms, with an adequate manipulation it is possible to obtain a decomposition similar to decomposition (3). In Proposition 4.2 we show that, when we restrict to $k = 1$, this decomposition is equivalent to the application of a "canonical splitting"-like algorithm to the corresponding variational morphism.

Using this fact we can finally compare, by Proposition 4.4, this "canonical splitting"-like decomposition with the decomposition (7), showing that the difference between the corresponding boundary terms is compensated by the difference between the corresponding volume terms.

3.1 Contact forms as variational morphisms

Consider an arbitrary bundle $\pi\colon Y \longrightarrow X$ with $n = \dim X$. Let $U \subseteq X$ be an open subset and let $W = \pi^{-1}(U)$ be the "tube" over U. Consider $\rho \in \Omega^r_q W$ a 1-contact q-form on $W^r = (\pi^r_0)^{-1}(W)$, with $q \leq n+1$. Then, if (W^r, ψ^r) is a local chart on $J^r Y$ associated with the fibered chart (W, ψ), $\psi = (x^i, y^\sigma)$ on Y, we can write

$$p_1\rho = \sum_{|J|=0}^{r} \omega^\sigma_J \wedge \eta^J_\sigma \ \in \Omega^{r+1}_q W$$

where η^J_σ are horizontal $(q-1)$-forms defined on W^{r+1} and thence can be expressed as

$$\eta^J_\sigma = A^{i_1\ldots i_s J}_\sigma (J^{r+1}y) ds_{i_1\ldots i_s}, \qquad s = n - (q-1).$$

Now, considering the vector bundle $V(W)$ whose sections are vertical vector fields defined over W and recalling that $J^r V(W) \cong V(J^r W)$, we can define according to Definition 2.2 a variational morphism $\mathbb{V}_\rho \colon J^{r+1}W \longrightarrow (J^r V(W))^* \otimes A_s(U)$ such that:

$$
\begin{aligned}
< \mathbb{V}_\rho | J^r \Xi >= J^r\Xi \lrcorner p_1\rho &= \Big[\sum_{|J|=0}^{r} A_\sigma^{i_1\dots i_s J} \Xi_J^\sigma \Big] \otimes ds_{i_1\dots i_s} = \\
&= \Big[A_\sigma^{i_1\dots i_s} \Xi^\sigma + A_\sigma^{i_1\dots i_s j_1} \Xi_{j_1}^\sigma + \dots + A_\sigma^{i_1\dots i_s j_1\dots j_r} \Xi_{j_1\dots j_r}^\sigma \Big] \otimes ds_{i_1\dots i_s}
\end{aligned}
$$

for every vertical vector field $\Xi \colon W \longrightarrow V(W)$.

The advantage of this approach consists in the possibility of working on contact forms (although only in the particular case of 1-contact forms of degree at most $n+1$) using the tools of the theory of variational morphisms and returning back to forms at the end of the manipulation.

It appears that the above identification of 1-contact forms with variational morphisms holds true up to $(n+1)$-forms and, indeed, this could be related to the non-uniqueness of the *source forms* providing the so-called *Helmholtz conditions*. In fact, as discussed in [18] (pag. 32), this feature appears for k-contact n-horizontal $(n+k)$-forms with $k \geq 2$.

3.2 Comparison for top forms

In this section we directly compare the two integration by parts procedures.

As a first step, in the following proposition we prove that the splitting (5) of $p_1\rho$, seen as a variational morphism $\mathbb{V}_\rho$ of codegree $s=0$,

$$< \mathbb{V}_\rho | J^r \Xi >=< \mathbb{E} | \Xi > + \mathrm{Div}(< \mathbb{T} | J^{r-1} \Xi >)$$

and the decomposition

$$p_1\rho = \mathcal{I}(\rho) + d_H \mathcal{R}(\rho)$$

give the same terms.

Proposition 3.1. *Given $\rho \in \Omega^r_{n+1} W$ a 1-contact $(n+1)$-form. For every section $\Xi \colon W \longrightarrow V(W)$,*

$$< \mathbb{V}_\rho | J^r \Xi >= J^r\Xi \lrcorner p_1\rho$$

and

$$< \mathbb{E} | \Xi >= J^r\Xi \lrcorner \mathcal{I}(\rho)\,, \quad Div(< \mathbb{T} | J^{r-1} \Xi >) = J^r\Xi \lrcorner\, d_H \mathcal{R}(\rho).$$

Proof. Step 1. The fact that $< \mathbb{E}|\Xi >= J^r\Xi \lrcorner \mathcal{I}(\rho)$ follows directly from the definition of each side of the equation. In fact, since W is a single coordinate domain, we can always choose the fibered connection whose coefficients are all null, then the covariant derivatives reduce to total derivatives and the variational morphism $\mathbb{E}$ takes the form:

$$< \mathbb{E}|\Xi >= [(A_\sigma - d_{j_1}A_\sigma^{j_1} + \cdots + (-1)^r d_{j_1\ldots j_r}A_\sigma^{j_1\ldots j_r})\Xi^\sigma] \otimes ds.$$

On the other hand, from the definition of the interior Euler operator, we have

$$\begin{aligned}\mathcal{I}(\rho) &= \omega^\sigma \wedge \sum_{|J|=0}^{r} (-1)^{|J|} \mathrm{d}_J \eta_\sigma^J = \omega^\sigma \wedge \Big(\sum_{|J|=0}^{r} (-1)^{|J|} d_J A_\sigma^J\Big) ds = \\ &= \omega^\sigma \wedge (A_\sigma - d_{j_1}A_\sigma^{j_1} + \cdots + (-1)^r d_{j_1\ldots j_r}A_\sigma^{j_1\ldots j_r})ds.\end{aligned}$$

Step 2. In order to compare $J^r\Xi \lrcorner\, d_H\mathcal{R}(\rho)$ with $\mathrm{Div}(< \mathbb{T}|J^{r-1}\Xi >)$ we need to compute explicitly the Residual operator $\mathcal{R}(\rho)$. We now write $p_1\rho = \sum_{|I|=0}^r \mathrm{d}_I\big(\omega^\sigma \wedge \xi_\sigma^I\big)$, where

$$\xi_\sigma^I = \sum_{|J|=0}^{r-|I|} (-1)^{|J|} \binom{|J|+|I|}{|J|} \mathrm{d}_J \eta_\sigma^{IJ} = \Bigg(\sum_{|J|=0}^{r-|I|} (-1)^{|J|} \binom{|J|+|I|}{|J|} \mathrm{d}_J A_\sigma^{IJ}\Bigg) ds\,.$$

The summand $\omega^\sigma \wedge \xi_\sigma$ is the interior Euler operator, so we consider only the remaining terms $\sum_{|I|=1}^r \mathrm{d}_I\big(\omega^\sigma \wedge \xi_\sigma^I\big)$. Each form $\omega^\sigma \wedge \xi_\sigma^I$ is a 1-contact $(n+1)$-form and thence can be recast as $\omega^\sigma \wedge \xi_\sigma^I = \chi^I \wedge ds$, where χ^I is a 1-contact 1-form locally given as

$$\chi^I = (\sum_{|J|=0}^{r-|I|} (-1)^{|J|} \binom{|J|+|I|}{|J|} \mathrm{d}_J A_\sigma^{IJ})\omega^\sigma.$$

Finally, the Residual operator is defined by

$$\begin{aligned}\mathcal{R}(\rho) &= \sum_{|I|=0}^{r-1} (-1)^1 \mathrm{d}_I \chi^{iI} \wedge ds_i = \\ &= \sum_{|I|=0}^{r-1} -\mathrm{d}_I \Bigg(\sum_{|J|=0}^{r-|iI|} (-1)^{|J|} \binom{|J|+|iI|}{|J|} \omega^\sigma \mathrm{d}_J A_\sigma^{iIJ}\Bigg) \wedge ds_i.\end{aligned}$$

Now, from the second equation above, we compute the coefficients of the forms $\omega_L^\sigma \wedge ds_i$ according to the length of the multi-index L:

$|L| = r-1$. The only contribution to this coefficient comes from setting $|I| = r-1$ and applying all the total derivatives of forms d_I to ω^σ, thus obtaining as coefficient A_σ^{iL}.

$|L| = r-2$. One contribution comes from setting $|I| = r-2$ and applying all the total derivatives of forms d_I to ω^σ, getting the coefficient

$$-(A_\sigma^{iL} - r d_l A_\sigma^{liL}),$$

another contribution comes from setting $|I| = r-1$ and applying $r-2$ total derivatives to ω^σ and one to A_σ^{iI}, thus obtaining

$$-\binom{r-1}{1} d_l A_\sigma^{liL},$$

summing together these two terms, we get the coefficient

$$-(A_\sigma^{iL} - d_l A_\sigma^{liL}).$$

$|L| = r-3$. We can take $|I| = r-3$ and apply all the total derivatives to ω^σ, getting a term

$$-A_\sigma^{iL} + \binom{r-1}{1} d_l A_\sigma^{liL} - \binom{r}{2} d_{lk} A_\sigma^{lkiL}\,,$$

another contribution comes from setting $|I| = r-2$ and applying only $r-3$ derivatives to ω^σ, getting a term

$$-\binom{r-2}{1} d_l A_\sigma^{liL} + \binom{r}{1}\binom{r-2}{1} d_{lk} A_\sigma^{lkiL}\,,$$

finally, we can take $|I| = r-1$ and apply $r-3$ derivatives to ω^σ, getting a term

$$-\binom{r-1}{2} d_{lk} A_\sigma^{lkiL},$$

summing all together these contributions we obtain the following coefficient

$$-(A_\sigma^{iL} - d_l A_\sigma^{liL} + d_{lk} A_\sigma^{lkiL})\,.$$

...

It appears clear that the coefficients of the forms $\omega_L^\sigma \wedge ds_i$ are defined by the same recurrence relations (6) which express the components t_σ^{iJ} of the variational morphism $\mathbb{T}$, except for the sign. Thence we can write

$$\mathcal{R}(\rho) = \sum_{|J|=0}^{r-1} -t_\sigma^{iJ}\omega_J^\sigma \wedge ds_i .$$

Therefore, $d_H\mathcal{R}(\rho) = p_1 dp_1\mathcal{R}(\rho) = \sum_{|J|=0}^{r-1}(d_i t_\sigma^{iJ}\omega_J^\sigma \wedge ds + t_\sigma^{iJ}\omega_{Ji}^\sigma \wedge ds)$, and since $\Xi_{Ji}^\sigma = d_i\Xi_J^\sigma$, we finally get

$$J^r\Xi \lrcorner\, d_H\mathcal{R}(\rho) = \sum_{|J|=0}^{r-1} \Big(d_i t_\sigma^{iJ}\Xi_J^\sigma + t_\sigma^{iJ}\Xi_{Ji}^\sigma \Big) \wedge ds =$$

$$= \sum_{|J|=0}^{r-1} \Big(d_i t_\sigma^{iJ}\Xi_J^\sigma + t_\sigma^{iJ} d_i\Xi_J^\sigma \Big) \wedge ds$$

$$= d_i\Big(\sum_{|J|=0}^{r-1} t_\sigma^{iJ}\Xi_J^\sigma \Big) \wedge ds = \mathrm{Div}(< \mathbb{T}|J^{r-1}\Xi >).$$

q.e.d.

Since $d_H ds_{i_1\dots i_s} = 0$, the latter result can be generalized to hold true for every $0 \leq s \leq n$; we can then state the following.

Proposition 3.2. *Let* $\mathbb{T}\colon J^tY \longrightarrow (J^{r-1}V(Y))^* \otimes A_{n-s}(X)$ *be a variational morphism according to Definition 2.2, with*

$$< \mathbb{T}|J^{r-1}\Xi >= (\sum_{|J|=0}^{r-1} t_\sigma^{i_1\dots i_s J}\Xi_J^\sigma) \wedge ds_{i_1\dots i_s}$$

for any vertical vector field Ξ *over* Y*. Then there is a correspondence between* $\mathbb{T}$ *and a* $(n-s)$*-horizontal* 1*-contact* $(n-s+1)$*-form* $\tilde{\mathcal{R}}$ *such that*

$$Div(< \mathbb{T}|J^{r-1}\Xi >) = J^r\Xi \lrcorner\, d_H\tilde{\mathcal{R}}$$

where $\tilde{\mathcal{R}}$ *is defined by*

$$\tilde{\mathcal{R}} = (\sum_{|J|=0}^{r-1} -t_\sigma^{i_1\dots i_s J}\omega_J^\sigma) \wedge ds_{i_1\dots i_s} .$$

4 The main result: comparison for lower degree forms

Consider a fiber bundle $\pi\colon Y \longrightarrow X$, $U \subseteq X$ an open subset and $W = \pi^{-1}(U)$. Let $\rho \in \Omega^r_{n-s+k}W$ be a $(n-s)$-horizontal k-contact $(n-s+k)$-form defined on the r-order jet prolongation W^r of W. In a local fibered chart $\psi^r = (x^i, y^\sigma, y^\sigma_I)$, we can write:

$$p_k\rho = \sum_{|J|=0}^{r} \omega^\sigma_J \wedge \eta^J_\sigma \ \in \Omega^{r+1}_{n-s+k}W$$

where η^J_σ are local $(n-s)$-horizontal $(k-1)$-contact $(n-s+k-1)$-forms defined on W^{r+1}. We can write

$$p_k\rho = \sum_{|J|=0}^{r} \omega^\sigma_J \wedge \eta^J_\sigma = \sum_{|I|=0}^{r} \mathrm{d}_I\big(\omega^\sigma \wedge \xi^I_\sigma\big) = \omega^\sigma \wedge \xi_\sigma + \sum_{|I|=1}^{r} \mathrm{d}_I\big(\omega^\sigma \wedge \xi^I_\sigma\big)$$

where

$$\xi^I_\sigma = \sum_{|J|=0}^{r-|I|} (-1)^{|J|} \binom{|J|+|I|}{|J|} d_J \eta^{IJ}_\sigma.$$

As we did for the case $s = 0$, we concentrate our attention on the term $\sum_{|I|=1}^{r} \mathrm{d}_I\big(\omega^\sigma \wedge \xi^I_\sigma\big)$. Each form $\omega^\sigma \wedge \xi^I_\sigma$ is a $(n-s)$-horizontal k-contact $(n-s+k)$-form, thus it can be recast in the following manner:

$$\omega^\sigma \wedge \xi^I_\sigma = \chi^{i_1\dots i_s I} \wedge ds_{i_1\dots i_s} \tag{9}$$

where the $\chi^{i_1\dots i_s I}$ are local k-contact k-forms. Using equation (9), renaming the multi-index I and extracting an antisymmetric part, we obtain:

$$\begin{aligned}
&\sum_{|I|=1}^{r} \mathrm{d}_I\big(\omega^\sigma \wedge \xi^I_\sigma\big) = \qquad (10)\\
&= \sum_{|I|=1}^{r} \mathrm{d}_I \chi^{i_1\dots i_s I} \wedge ds_{i_1\dots i_s} = \sum_{|I|=0}^{r-1} \mathrm{d}_i \mathrm{d}_I \chi^{i_1\dots i_s i I} \wedge ds_{i_1\dots i_s} = \\
&= \sum_{|I|=0}^{r-1} \mathrm{d}_i \mathrm{d}_I \Big(\chi^{i_1\dots i_s i I} - \chi^{[i_1\dots i_s i]I}\Big) \wedge ds_{i_1\dots i_s} + \mathrm{d}_i \sum_{|I|=0}^{r-1} \mathrm{d}_I \chi^{[i_1\dots i_s i]I} \wedge ds_{i_1\dots i_s}.
\end{aligned}$$

In the following we show that the local form

$$\mathrm{d}_i \sum_{|I|=0}^{r-1} \mathrm{d}_I \chi^{[i_1 \dots i_s i] I} \wedge ds_{i_1 \dots i_s} \tag{11}$$

can be expressed as the horizontal differential of a $(n-s-1)$-horizontal k-contact $(n-s-1+k)$-form, obtained by the application of a suitably defined Residual operator for lower degree forms $\mathscr{R}$.

Proposition 4.1. *Let $\rho \in \Omega^r_{n-s+k} W$ be a $(n-s)$-horizontal k-contact $(n-s+k)$-form defined on the r-order jet prolongation W^r of W. We have*

$$\begin{aligned}
&d_i \sum_{|I|=0}^{r-1} d_I \chi^{[i_1 \dots i_s i] I} \wedge ds_{i_1 \dots i_s} = \\
&d_H \Bigg(\sum_{|I|=0}^{r-1} (-1)^k \frac{1}{(s+1)} d_I \chi^{[i_1 \dots i_s i] I} \wedge ds_{i_1 \dots i_s i} \Bigg) = d_H \mathscr{R}(\rho).
\end{aligned}$$

Proof. First we rewrite expression (11) using a summation over *ordered* indices $\tilde{i}_1 \leq \dots \leq \tilde{i}_s$ instead of $i_1 \dots i_s$:

$$\mathrm{d}_i \sum_{|I|=0}^{r-1} \mathrm{d}_I \chi^{[i_1 \dots i_s i] I} \wedge ds_{i_1 \dots i_s} = \mathrm{d}_i \sum_{|I|=0}^{r-1} s! \, \mathrm{d}_I \chi^{[\tilde{i}_1 \dots \tilde{i}_s i] I} \wedge ds_{\tilde{i}_1 \dots \tilde{i}_s}.$$

Then we expand this sum, using Einstein's convention for the summation over multi-indices I:

$$\begin{aligned}
&\mathrm{d}_i \sum_{|I|=0}^{r-1} s! \, \mathrm{d}_I \chi^{[\tilde{i}_1 \dots \tilde{i}_s i] I} \wedge ds_{\tilde{i}_1 \dots \tilde{i}_s} = \\
&= s! \Big(\mathrm{d}_1 \mathrm{d}_I \chi^{[\tilde{i}_1 \dots \tilde{i}_s 1] I} \wedge ds_{\tilde{i}_1 \dots \tilde{i}_s} + \mathrm{d}_2 \mathrm{d}_I \chi^{[\tilde{i}_1 \dots \tilde{i}_s 2] I} \wedge ds_{\tilde{i}_1 \dots \tilde{i}_s} + \dots \\
&\dots + \mathrm{d}_n \mathrm{d}_I \chi^{[\tilde{i}_1 \dots \tilde{i}_s n] I} \wedge ds_{\tilde{i}_1 \dots \tilde{i}_s} \Big).
\end{aligned}$$

The following step towards the thesis is to observe that

$$ds_{\tilde{i}_1 \dots \tilde{i}_s} = dx^i \wedge ds_{\tilde{i}_1 \dots \tilde{i}_s i}$$

without summation on the index i and with $i \neq \tilde{i}_1, \ldots, \tilde{i}_s$. Then we can proceed as follows:

$$
\begin{aligned}
&\mathrm{d}_i \sum_{|I|=0}^{r-1} s!\, \mathrm{d}_I \chi^{[\tilde{i}_1 \ldots \tilde{i}_s i]I} \wedge ds_{\tilde{i}_1 \ldots \tilde{i}_s} = \qquad (12)\\
&= s! \Big(\mathrm{d}_1 \mathrm{d}_I \chi^{[\tilde{i}_1 \ldots \tilde{i}_s 1]I} \wedge dx^1 \wedge ds_{\tilde{i}_1 \ldots \tilde{i}_s 1} + \mathrm{d}_2 \mathrm{d}_I \chi^{[\tilde{i}_1 \ldots \tilde{i}_s 2]I} \wedge dx^2 \wedge ds_{\tilde{i}_1 \ldots \tilde{i}_s 2} + \ldots \\
&\cdots + \mathrm{d}_n \mathrm{d}_I \chi^{[\tilde{i}_1 \ldots \tilde{i}_s n]I} \wedge dx^n \wedge ds_{\tilde{i}_1 \ldots \tilde{i}_s n} \Big) .
\end{aligned}
$$

For each term of the last equation, we have the same indices in $\chi^{[\tilde{i}_1 \ldots \tilde{i}_s i]I}$ and in $ds_{\tilde{i}_1 \ldots \tilde{i}_s i}$, though without summation on i.

Our goal is to take each term $\mathrm{d}_i \mathrm{d}_I \chi^{[\tilde{i}_1 \ldots \tilde{i}_s i]I} \wedge dx^i \wedge ds_{\tilde{i}_1 \ldots \tilde{i}_s i}$ without summation on any index and find a way to write it in the form

$$\mathrm{d}_l \mathrm{d}_I \chi^{[\tilde{i}_1 \ldots \tilde{i}_s i]I} \wedge dx^l \wedge ds_{\tilde{i}_1 \ldots \tilde{i}_s i}$$

with summation on the index l. In order to make explicit the reasoning, let us consider the term with $i = 1$

$$\mathrm{d}_1 \mathrm{d}_I \chi^{[\tilde{i}_1 \ldots \tilde{i}_s 1]I} \wedge dx^1 \wedge ds_{\tilde{i}_1 \ldots \tilde{i}_s 1}$$

again without summation over any index. We have obviously that:

$$\mathrm{d}_1 \mathrm{d}_I \chi^{[\tilde{i}_1 \ldots \tilde{i}_s 1]I} \wedge dx^1 \wedge ds_{\tilde{i}_1 \ldots \tilde{i}_s 1} = \sum_{l \neq \tilde{i}_1, \ldots, \tilde{i}_s} \mathrm{d}_l \mathrm{d}_I \chi^{[\tilde{i}_1 \ldots \tilde{i}_s 1]I} \wedge dx^l \wedge ds_{\tilde{i}_1 \ldots \tilde{i}_s 1}.$$

The remaining terms when $l = \tilde{i}_1, \ldots, l = \tilde{i}_s$ can be found among the other summands of the right hand side of equation (12). As an example, consider the case when $l = \tilde{i}_1$: among the terms

$$\mathrm{d}_{\tilde{i}_1} \mathrm{d}_I \chi^{[\tilde{j}_1 \ldots \tilde{j}_s \tilde{i}_1]I} \wedge dx^{\tilde{i}_1} \wedge ds_{\tilde{j}_1 \ldots \tilde{j}_s \tilde{i}_1}$$

(with summation on the ordered indices $\tilde{j}_1 \leq \cdots \leq \tilde{j}_s$) there will certainly be a summand of the form

$$\mathrm{d}_{\tilde{i}_1} \mathrm{d}_I \chi^{[1 \tilde{i}_2 \ldots \tilde{i}_s \tilde{i}_1]I} \wedge dx^{\tilde{i}_1} \wedge ds_{1 \tilde{i}_2 \ldots \tilde{i}_s \tilde{i}_1}$$

(this time without summation on any index) which can be recast as

$$\mathrm{d}_{\tilde{i}_1} \mathrm{d}_I \chi^{[\tilde{i}_1 \ldots \tilde{i}_s 1]I} \wedge dx^{\tilde{i}_1} \wedge ds_{\tilde{i}_1 \ldots \tilde{i}_s 1}.$$

This is exactly the term we were searching for. Proceeding the same way for $l = \tilde{i}_2, \dots, l = \tilde{i}_s$, we can finally obtain the expression

$$\mathrm{d}_l \mathrm{d}_I \chi^{[\tilde{i}_1 \dots \tilde{i}_s 1]I} \wedge dx^l \wedge ds_{\tilde{i}_1 \dots \tilde{i}_s 1}$$

(with summation on l) as wanted. Moreover, we remark that the total number of summands of equation (12) is $N = n\binom{n-1}{s}$, while the number of ordered strings of $(s+1)$ indices is $N' = \binom{n}{s+1}$, and $N = (s+1)N'$. This means that for every single term of equation (12) $\mathrm{d}_i \mathrm{d}_I \chi^{[\tilde{i}_1 \dots \tilde{i}_s i]I} \wedge dx^i \wedge ds_{\tilde{i}_1 \dots \tilde{i}_s i}$ (without summation on any index), there are other s terms with the same indices, just in a different order.

Therefore, it is not difficult to see that

$$\mathrm{d}_i \sum_{|I|=0}^{r-1} s!\, \mathrm{d}_I \chi^{[\tilde{i}_1 \dots \tilde{i}_s i]I} \wedge ds_{\tilde{i}_1 \dots \tilde{i}_s} = \mathrm{d}_l \sum_{|I|=0}^{r-1} s!\, \mathrm{d}_I \chi^{[\tilde{i}_1 \dots \tilde{i}_s \tilde{i}_{s+1}]I} \wedge dx^l \wedge ds_{\tilde{i}_1 \dots \tilde{i}_s \tilde{i}_{s+1}}$$

with summation on ordered indices $\tilde{i}_1 \leq \dots \leq \tilde{i}_s \leq \tilde{i}_{s+1}$. Passing to a summation on non-ordered indices $i_1 \dots i_{s+1}$ and using commutation properties of wedge products, we can finally write:

$$\begin{aligned}
&\mathrm{d}_i \sum_{|I|=0}^{r-1} \mathrm{d}_I \chi^{[i_1 \dots i_s i]I} \wedge ds_{i_1 \dots i_s} = \\
&= \mathrm{d}_l \sum_{|I|=0}^{r-1} \frac{1}{(s+1)} \mathrm{d}_I \chi^{[i_1 \dots i_{s+1}]I} \wedge dx^l \wedge ds_{i_1 \dots i_{s+1}} = \\
&= (-1)^{n-s-1+k} \mathrm{d}_l \left(\sum_{|I|=0}^{r-1} (-1)^k \frac{1}{(s+1)} \mathrm{d}_I \chi^{[i_1 \dots i_{s+1}]I} \wedge ds_{i_1 \dots i_{s+1}} \right) \wedge dx^l = \\
&= d_H \left(\sum_{|I|=0}^{r-1} (-1)^k \frac{1}{(s+1)} \mathrm{d}_I \chi^{[i_1 \dots i_{s+1}]I} \wedge ds_{i_1 \dots i_{s+1}} \right) = d_H \mathscr{R}(\rho).
\end{aligned}$$

q.e.d.

Now, in order to compare the decomposition of equation (10) with the canonical splitting of variational morphisms, we can state the following result:

Proposition 4.2. *Restricting to the particular case $k = 1$, we have that the decomposition*

$$p_1\rho = \omega^\sigma \wedge \xi_\sigma + \sum_{|I|=0}^{r-1} d_i d_I(\chi^{i_1\dots i_s iI} - \chi^{[i_1\dots i_s i]I}) \wedge ds_{i_1\dots i_s} +$$
$$+d_i \sum_{|I|=0}^{r-1} d_I \chi^{[i_1\dots i_s i]I} \wedge ds_{i_1\dots i_s} \tag{13}$$

can be seen as the result of the application of a "canonical splitting"-like algorithm to the variational morphism associated to $p_1\rho$.

Proof. First of all we remark that when $k = 1$, we have:

$$\eta^J_\sigma = A^{i_1\dots i_s J}_\sigma (J^{r+1}y) \wedge ds_{i_1\dots i_s},$$

thence

$$\xi^I_\sigma = (\sum_{|J|=0}^{r-|I|} (-1)^{|J|} \binom{|J|+|I|}{|J|} d_J A^{i_1\dots i_s IJ}_\sigma) ds_{i_1\dots i_s}$$

and consequently

$$\chi^{i_1\dots i_s I} = \sum_{|J|=0}^{r-|I|} (-1)^{|J|} \binom{|J|+|I|}{|J|} d_J A^{i_1\dots i_s IJ}_\sigma \omega^\sigma. \tag{14}$$

We remark that the indices $i_1 \dots i_s$, which in the last two expressions are contracted with the horizontal form $ds_{i_1\dots i_s}$ have nothing to do with the multi-index I.

By Proposition 4.1 and equation (14) we have that the third summand of equation (13) can be expressed as the horizontal differential of a $(n-s-1)$-horizontal 1-contact $(n-s)$-form, in particular:

$$\mathrm{d}_i \sum_{|I|=0}^{r-1} \mathrm{d}_I \chi^{[i_1\dots i_s i]I} \wedge ds_{i_1\dots i_s} = \tag{15}$$
$$= d_H[\sum_{|I|=0}^{r-1} \frac{-1}{(s+1)} \mathrm{d}_I \chi^{[i_1\dots i_s i]I} \wedge ds_{i_1\dots i_s i}] =$$
$$= d_H[\frac{-1}{(s+1)} \sum_{|I|=0}^{r-1} \mathrm{d}_I (\sum_{|J|=0}^{r-|iI|} (-1)^{|J|} \binom{|J|+|iI|}{|J|} d_J A^{[i_1\dots i_s i]IJ}_\sigma \omega^\sigma) \wedge ds_{i_1\dots i_s i}].$$

According to Proposition 3.2 this term corresponds to the divergence term of the "canonical splitting"-like algorithm. Its explicit form can be computed as described in *Step 2* of the proof of Proposition 3.1. In particular, if we develop the total derivatives d_I inside the sum in equation (15) and collect the coefficients of the contact forms of the same order, we obtain the expression

$$d_H\left[\frac{-1}{(s+1)}\sum_{|I|=0}^{r-1}\mathrm{d}_I\Big(\sum_{|J|=0}^{r-|iI|}(-1)^{|J|}\binom{|J|+|iI|}{|J|}d_J A_\sigma^{[i_1\dots i_s i]IJ}\omega^\sigma\Big)\wedge ds_{i_1\dots i_s i}\right]=$$

$$= d_H[-\frac{1}{(s+1)}\sum_{|L|=0}^{r-1}\hat{t}_\sigma^{i_1\dots i_s iL}\omega_L^\sigma\wedge ds_{i_1\dots i_s i}] \tag{16}$$

where the coefficients $\hat{t}$'s are defined iteratively by:

$$\begin{aligned}
&\hat{t}_\sigma^{i_1\dots i_s i l_1\dots l_{r-1}} = A_\sigma^{[i_1\dots i_s i]l_1\dots l_{r-1}}\\
&\hat{t}_\sigma^{i_1\dots i_s i l_1\dots l_{r-2}} = A_\sigma^{[i_1\dots i_s i]l_1\dots l_{r-2}} - d_k\hat{t}_\sigma^{i_1\dots i_s i l_1\dots l_{r-2}k}\\
&\dots\\
&\hat{t}_\sigma^{i_1\dots i_s i} = A_\sigma^{[i_1\dots i_s i]} - d_k\hat{t}_\sigma^{i_1\dots i_s ik}\,.
\end{aligned} \tag{17}$$

In order to understand how the algorithm is defined, we have to compute explicitly the other terms of decomposition (13). Let us start, using equation (14), with:

$$\chi^{i_1\dots i_s iI}-\chi^{[i_1\dots i_s i]I}=\sum_{|J|=0}^{r-|iI|}(-1)^{|J|}\binom{|J|+|iI|}{|J|}d_J\Big(A_\sigma^{i_1\dots i_s iIJ}-A_\sigma^{[i_1\dots i_s i]IJ}\Big)\omega^\sigma.$$

We call $B_\sigma^{i_1\dots i_s iIJ} = A_\sigma^{i_1\dots i_s iIJ} - A_\sigma^{[i_1\dots i_s i]IJ}$ and remark that this object is antisymmetric in the indices $i_1\dots i_s$, symmetric in the indices of the multi-indices $IJ := (\hat{i}_1\dots\hat{i}_l\hat{j}_1\dots\hat{j}_m)$, but it is not completely symmetric in $iI := (i\hat{i}_1\dots\hat{i}_l)$.

So, what we want to make explicit can be written as:

$$\sum_{|I|=0}^{r-1}\mathrm{d}_i\mathrm{d}_I\Big(\chi^{i_1\dots i_s iI}-\chi^{[i_1\dots i_s i]I}\Big)\wedge ds_{i_1\dots i_s}= \tag{18}$$

$$=\sum_{|I|=0}^{r-1}\mathrm{d}_i\mathrm{d}_I\Big(\sum_{|J|=0}^{r-|iI|}(-1)^{|J|}\binom{|J|+|iI|}{|J|}d_J B_\sigma^{i_1\dots i_s iIJ}\omega^\sigma\Big)\wedge ds_{i_1\dots i_s}\,.$$

Developing the total derivatives of forms of this expression, we obtain a linear combination of the forms $\omega^\sigma_L \wedge ds_{i_1\dots i_s}$. We compute the coefficients of this linear combination according to the length of the multi-index L:

$|L| = r$ The only contribution to this coefficient comes from setting $|I| = r-1$ in equation (18) and applying all the total derivatives to ω^σ. Renaming the multi-index $L = lL'$ (hence with $|L'| = |L| - 1$), we obtain:

$$B^{i_1\dots i_s lL'}_\sigma = A^{i_1\dots i_s lL'}_\sigma - A^{[i_1\dots i_s l]L'}_\sigma .$$

$|L| = r-1$ One first contribution comes from setting $|I| = r-2$ and applying all the total derivatives to ω^σ, getting:

$$B^{i_1\dots i_s L}_\sigma - r d_i B^{i_1\dots i_s Li}_\sigma .$$

Another contribution comes from setting $|I| = r-1$ and applying one derivative to B and the others to ω^σ, getting:

$$d_i B^{i_1\dots i_s iL}_\sigma + \binom{r-1}{1} d_i B^{i_1\dots i_s Li}_\sigma .$$

Summing up these contributions, and renaming $L = lL'$, we obtain:

$$\begin{aligned}
&B^{i_1\dots i_s lL'}_\sigma - d_i B^{i_1\dots i_s lL'i}_\sigma + d_i B^{i_1\dots i_s ilL'}_\sigma = \\
&= A^{i_1\dots i_s lL'}_\sigma - A^{[i_1\dots i_s l]L'}_\sigma - d_i A^{i_1\dots i_s lL'i}_\sigma + d_i A^{[i_1\dots i_s l]L'i}_\sigma + \\
&+ d_i A^{i_1\dots i_s ilL'}_\sigma - d_i A^{[i_1\dots i_s i]lL'}_\sigma = \\
&= A^{i_1\dots i_s lL'}_\sigma - A^{[i_1\dots i_s l]L'}_\sigma + d_i A^{[i_1\dots i_s l]L'i}_\sigma - d_i A^{[i_1\dots i_s i]lL'}_\sigma .
\end{aligned}$$

$|L| = r-2$ One contribution comes from setting $|I| = r-3$ and applying all the derivatives to ω^σ, getting a term (again we rename $L = lL'$):

$$B^{i_1\dots i_s lL'}_\sigma - \binom{r-1}{1} d_i B^{i_1\dots i_s lL'i}_\sigma + \binom{r}{2} d_{ab} B^{i_1\dots i_s lL'ab}_\sigma .$$

Another contribution comes from setting $|I| = r-2$ and applying only $r-2$ derivatives to ω^σ, getting:

$$\begin{aligned}
&d_i B^{i_1\dots i_s ilL'}_\sigma + \binom{r-2}{1} d_i B^{i_1\dots i_s lL'i}_\sigma - \\
&- r d_{ia} B^{i_1\dots i_s ialL'}_\sigma - r\binom{r-2}{1} d_{ab} B^{i_1\dots i_s lL'ab}_\sigma .
\end{aligned}$$

One last contribution comes from setting $|I| = r - 1$ and applying only $r - 2$ derivatives to ω^σ, getting:

$$\binom{r-1}{1} d_{ia} B_\sigma^{i_1 \ldots i_s i a l L'} + \binom{r-1}{2} d_{ab} B_\sigma^{i_1 \ldots i_s l L' ab}.$$

Summing first the terms with two total derivatives, we obtain:

$$-d_{ia} B_\sigma^{i_1 \ldots i_s i a l L'} + d_{ab} B_\sigma^{i_1 \ldots i_s l L' ab} = d_{ia} A_\sigma^{[i_1 \ldots i_s i] a l L'} - d_{ab} A_\sigma^{[i_1 \ldots i_s l] L' ab}.$$

Summing the terms with one total derivative of the B's, we obtain:

$$d_i B_\sigma^{i_1 \ldots i_s i l L'} - d_i B_\sigma^{i_1 \ldots i_s l L' i} = -d_i A_\sigma^{[i_1 \ldots i_s i] l L'} + d_i A_\sigma^{[i_1 \ldots i_s l] L' i}$$

Summing all together, we finally obtain that the coefficient for $|L| = r - 2$ is:

$$\begin{aligned} &A_\sigma^{i_1 \ldots i_s l L'} - A_\sigma^{[i_1 \ldots i_s l] L'} + d_i A_\sigma^{[i_1 \ldots i_s l] L' i} - d_{ab} A_\sigma^{[i_1 \ldots i_s l] L' ab} - \\ &\quad - d_i A_\sigma^{[i_1 \ldots i_s i] l L'} d_{ia} A_\sigma^{[i_1 \ldots i_s i] a l L'} \end{aligned}$$

...

It remains to see what is the coefficient for $|L| = 0$, i.e. the coefficient of the form $\omega^\sigma \wedge ds_{i_1 \ldots i_s}$. The first term we need to consider is obviously $\omega^\sigma \wedge \xi_\sigma$, where

$$\xi_\sigma = \left(\sum_{|J|=0}^{r} (-1)^{|J|} d_J A_\sigma^{i_1 \ldots i_s J} \right) \wedge ds_{i_1 \ldots i_s}.$$

The other contributions come from

$$\sum_{|I|=0}^{r-1} \mathrm{d}_i \mathrm{d}_I \left(\sum_{|J|=0}^{r-|iI|} (-1)^{|J|} \binom{|J| + |iI|}{|J|} d_J B_\sigma^{i_1 \ldots i_s i I J} \omega^\sigma \right) \wedge ds_{i_1 \ldots i_s}$$

applying all the total derivatives to the B's. In other words, renaming $iI \to I$, we want to compute the expression:

$$\sum_{|J|=0}^{r} (-1)^{|J|} d_J A_\sigma^{i_1 \ldots i_s J} + \sum_{|I|=1}^{r} \sum_{|J|=0}^{r-|I|} (-1)^{|J|} \binom{|J| + |I|}{|J|} d_I d_J B_\sigma^{i_1 \ldots i_s I J}. \tag{19}$$

Again, we collect the terms of equation (19) according to the length L of the multi-index J in the first sum and IJ in the second sum:

$L = 0$ Since the second double sum of equation (19) starts with $|I| = 1$, we have only the first term $A_\sigma^{i_1 \dots i_s}$ from the first sum.

$L = 1$ From the first sum we get the term $-d_i A_\sigma^{i_1 \dots i_s i}$, while from the second sum, considering $|I| = 1$ and $|J| = 0$, we get $d_i B_\sigma^{i_1 \dots i_s i}$. Summing together these contribution we obtain:

$$-d_i A_\sigma^{i_1 \dots i_s i} + d_i B_\sigma^{i_1 \dots i_s i} = -d_i A_\sigma^{[i_1 \dots i_s i]}.$$

$L = 2$ From the first sum we get the term $d_{ab} A_\sigma^{i_1 \dots i_s ab}$, while in the second sum we can choose $|I| = 2$ and $|J| = 0$, getting the term $d_{ab} B_\sigma^{i_1 \dots i_s ab}$, or $|I| = 1$ and $|J| = 1$, getting the term $-2 d_{ab} B_\sigma^{i_1 \dots i_s ab}$. Summing all together, we obtain:

$$d_{ab} A_\sigma^{i_1 \dots i_s ab} - d_{ab} B_\sigma^{i_1 \dots i_s ab} = d_a d_b A_\sigma^{[i_1 \dots i_s a]b}.$$

...

It is not difficult to see that, for general length L, from expression (19), we obtain the term:

$$\begin{aligned}
&(-1)^L d_{a_1 \dots a_L} A_\sigma^{i_1 \dots i_s a_1 \dots a_L} + \sum_{k=0}^{L-1} (-1)^k \binom{L}{k} d_{a_1 \dots a_L} B_\sigma^{i_1 \dots i_s a_1 \dots a_L} = \\
&= (-1)^L \Big(d_{a_1 \dots a_L} A_\sigma^{i_1 \dots i_s a_1 \dots a_L} - d_{a_1 \dots a_L} B_\sigma^{i_1 \dots i_s a_1 \dots a_L} \Big) = \\
&= (-1)^L d_{a_1 \dots a_L} A_\sigma^{[i_1 \dots i_s a_1] a_2 \dots a_L}.
\end{aligned}$$

where we used $\sum_{k=0}^{L-1} (-1)^k \binom{L}{k} = -(-1)^L$. Thence finally, the coefficient of the form $\omega^\sigma \wedge ds_{i_1 \dots i_s}$ is given by:

$$\begin{aligned}
A_\sigma^{i_1 \dots i_s} - d_{a_1} A_\sigma^{[i_1 \dots i_s a_1]} + d_{a_1} d_{a_2} A_\sigma^{[i_1 \dots i_s a_1] a_2} + \dots \\
\dots + (-1)^r d_{a_1} \dots d_{a_r} A_\sigma^{[i_1 \dots i_s a_1] a_2 \dots a_r}.
\end{aligned}$$

The form of the coefficients of the forms $\omega_L^\sigma \wedge ds_{i_1 \dots i_s}$, with $|L| = 0, 1, 2, \dots, r$, that we have computed from expression (18), suggests us the definition of a "canonical splitting"-like algorithm.

Let us consider the $(n-s)$-horizontal 1-contact $(n-s+1)$-form $p_1 \rho \in \Omega_{n-s+1}^{r+1} W$ as a variational morphism

$$\mathbb{V}_\rho \colon J^{r+1} W \longrightarrow (J^r V(W))^* \otimes A_{n-s}(U)$$

with

$$\begin{aligned}&< \mathbb{V}_\rho | J^r\Xi >:= J^r\Xi \lrcorner\, p_1\rho = \\ &= (A^{i_1\dots i_s}_\sigma \Xi^\sigma + A^{i_1\dots i_s j_1}_\sigma \Xi^\sigma_{j_1} + \dots + A^{i_1\dots i_s j_1\dots j_r}_\sigma \Xi^\sigma_{j_1\dots j_r}) \wedge ds_{i_1\dots i_s}.\end{aligned}$$

According to Propositions 3.1 and 3.2 and with respect to equation (13), we can define two variational morphisms

$$\begin{aligned}&\mathbb{E}' \colon J^{2r+1}W \longrightarrow (J^rV(W))^* \otimes A_{n-s}(U) \\ &\mathbb{T}' \colon J^{2r}W \longrightarrow (J^{r-1}V(W))^* \otimes A_{n-s-1}(U),\end{aligned}$$

such that

$$< \mathbb{V}_\rho | J^r\Xi > = < \mathbb{E}' | J^r\Xi > + \mathrm{Div}(< \mathbb{T}' | J^{r-1}\Xi >)$$

and in particular we have the correspondences:

$$< \mathbb{E}' | J^r\Xi > = J^r\Xi \lrcorner (\omega^\sigma \wedge \xi_\sigma + \sum_{|I|=0}^{r-1} \mathrm{d}_i \mathrm{d}_I (\chi^{i_1\dots i_s iI} - \chi^{[i_1\dots i_s i]I}) \wedge ds_{i_1\dots i_s}),$$

and

$$\mathrm{Div}(< \mathbb{T}' | J^{r-1}\Xi >) = J^r\Xi \lrcorner\, d_H (\sum_{|I|=0}^{r-1} - \frac{1}{(s+1)} \mathrm{d}_I \chi^{[i_1\dots i_s i]I} \wedge ds_{i_1\dots i_s i}).$$

Using equations (16) and (17) and Proposition 3.2 it is immediate to see that the components of $\mathbb{T}'$ are given by:

$$\begin{aligned}< \mathbb{T}' | J^{r-1}\Xi > = (&\hat{t}^{i_1\dots i_{s+1}}_\sigma \Xi^\sigma + \hat{t}^{i_1\dots i_{s+1} j_1}_\sigma \Xi^\sigma_{j_1} + \dots \\ &\dots + \hat{t}^{i_1\dots i_{s+1} j_1\dots j_{r-1}}_\sigma \Xi^\sigma_{j_1\dots j_{r-1}}) \wedge ds_{i_1\dots i_s i_{s+1}}.\end{aligned}$$

Denoting with $\hat{e}$ the components of $\mathbb{E}'$, so that

$$< \mathbb{E}' | J^r\Xi > = (\hat{e}^{i_1\dots i_s}_\sigma \Xi^\sigma + \hat{e}^{i_1\dots i_s j_1}_\sigma \Xi^\sigma_{j_1} + \dots + \hat{e}^{i_1\dots i_s j_1\dots j_r}_\sigma \Xi^\sigma_{j_1\dots j_r}) \wedge ds_{i_1\dots i_s}$$

and comparing with the results of the calculations described above, we can finally give the general form of the coefficients $\hat{e}$:

$$\begin{aligned}&\hat{e}^{i_1\dots i_s j_1\dots j_r}_\sigma = A^{i_1\dots i_s j_1\dots j_r}_\sigma - \hat{t}^{i_1\dots i_s j_1\dots j_r}_\sigma, \\ &\hat{e}^{i_1\dots i_s j_1\dots j_h}_\sigma = A^{i_1\dots i_s j_1\dots j_h}_\sigma - d_i \hat{t}^{i_1\dots i_s i j_1\dots j_h}_\sigma - \hat{t}^{i_1\dots i_s j_1\dots j_h}_\sigma \qquad 1 \le h < r, \\ &\hat{e}^{i_1\dots i_s}_\sigma = A^{i_1\dots i_s}_\sigma - d_i \hat{t}^{i_1\dots i_s i}_\sigma.\end{aligned}$$

q.e.d.

Remark 4.1. In the particular case of a form $\chi^{i_1\dots i_s iI}$ antisymmetric in the indices $[i_1 \dots i_s i]$ on its own, *i.e.* when $\chi^{i_1\dots i_s iI} = \chi^{[i_1\dots i_s i]I}$, we obtain a decomposition

$$p_1\rho = \omega^\sigma \wedge \xi_\sigma + d_H\Big[\sum_{|I|=0}^{r-1} -\frac{1}{(s+1)}\mathrm{d}_I\chi^{i_1\dots i_s iI} \wedge ds_{i_1\dots i_s i}\Big]$$

which first appeared in [18] and it formally resembles to the decomposition (3) obtained by Krbek and Musilová for top forms.

Remark 4.2. We used the term *"canonical splitting"-like algorithm* to describe the decomposition of Proposition 4.2 because the coefficients $\hat{e}$ of $\mathbb{E}'$ are not symmetric in the indices $(j_1 \dots j_h)$ (this is due to the presence of the terms $\hat{t}_\sigma^{i_1\dots i_s j_1\dots j_h}$ which are not symmetric in their last h indices) as they should. Moreover the variational morphism $\mathbb{E}'$ is not even reduced in the sense of Definition 2.3, while the volume part is reduced in that sense.

Indeed, we can give an account for the difference between the splitting

$$< \mathbb{V}_\rho | J^r\Xi >:= J^r\Xi \rfloor p_1\rho =< \mathbb{E}|J^r\Xi > +\mathrm{Div}(< \mathbb{T}|J^{r-1}\Xi >), \qquad (20)$$

with

$$\begin{aligned}
&\mathbb{V}_\rho\colon J^{r+1}W \longrightarrow (J^rV(W))^* \otimes A_{n-s}(U)\\
&\mathbb{E}\colon J^{2r+1}W \longrightarrow (J^rV(W))^* \otimes A_{n-s}(U)\\
&\mathbb{T}\colon J^{2r}W \longrightarrow (J^{r-1}V(W))^* \otimes A_{n-s-1}(U),
\end{aligned}$$

and the decomposition (13)

$$\begin{aligned}
J^r\Xi \rfloor\, p_1\rho = J^r\Xi \rfloor (\omega^\sigma \wedge \xi_\sigma + \sum_{|I|=0}^{r-1} \mathrm{d}_i\mathrm{d}_I\Big(\chi^{i_1\dots i_s iI} - \chi^{[i_1\dots i_s i]I}) \wedge ds_{i_1\dots i_s}\Big) + \\
+ J^r\Xi \rfloor\, d_H(\sum_{|I|=0}^{r-1} -\frac{1}{(s+1)}\mathrm{d}_I\chi^{[i_1\dots i_s i]I} \wedge ds_{i_1\dots i_s i})
\end{aligned}$$

which, as we have just shown, can be seen as a local variational morphism splitting

$$J^r\Xi \rfloor\, p_1\rho =:< \mathbb{V}_\rho|J^r\Xi >=< \mathbb{E}'|J^r\Xi > +\mathrm{Div}(< \mathbb{T}'|J^{r-1}\Xi >),$$

with

$$\mathbb{E}' \colon J^{2r+1}W \longrightarrow (J^r V(W))^* \otimes A_{n-s}(U)$$
$$\mathbb{T}' \colon J^{2r}W \longrightarrow (J^{r-1} V(W))^* \otimes A_{n-s-1}(U),$$

where the morphisms $\mathbb{E}'$ and $\mathbb{T}'$ are defined according to Proposition 4.2.
In particular, in view of possible applications to *momentum morphisms*, i.e. 1-contact $(n-1)$-forms (related to the Poincaré-Cartan morphism), we can state the following results (that we guess could be extended to the general case).

Proposition 4.3. *Let $r = 1$, $0 \leq s < n$ and let $\rho \in \Omega^1_{n-s+1} W$ be a 1-contact $(n-s)$-horizontal form on W^1. Then the two splitting formulae:*

$$< \mathbb{V}_\rho | J^1\Xi > = < \mathbb{E} | J^1 \Xi > + Div(< \mathbb{T} | \Xi >) = < \mathbb{E}' | J^1 \Xi > + Div(< \mathbb{T}' | \Xi >)$$

give the same result. In other words:

$$\mathbb{E}' = \mathbb{E}\,, \qquad \mathbb{T}' = \mathbb{T}.$$

Proof. We observe that locally we have

$$p_1 \rho = \big(A^{i_1 \ldots i_s}_\sigma \omega^\sigma + A^{i_1 \ldots i_s j}_\sigma \omega^\sigma_j \big) \wedge ds_{i_1 \ldots i_s} \in \Omega^2_{n-s+1} W.$$

Thence, for any vertical vector field $\Xi \colon W \longrightarrow V(W)$, the variational morphism $\mathbb{V}_\rho \colon J^2 W \longrightarrow \big(J^1 V(W)\big)^* \otimes A_{n-s}(U)$ is defined by:

$$< \mathbb{V}_\rho | J^1 \Xi > = \frac{1}{s!} \big(s! A^{i_1 \ldots i_s}_\sigma \Xi^\sigma + s! A^{i_1 \ldots i_s j}_\sigma \Xi^\sigma_j \big) \wedge ds_{i_1 \ldots i_s}.$$

Applying to this expression of $\mathbb{V}_\rho$ the canonical splitting algorithm from the theory of variational morphisms, we easily obtain the following volume term and boundary term, respectively:

$$\mathbb{E} = \frac{1}{s!} \big[s! \big(A^{i_1 \ldots i_s}_\sigma - d_k A^{[i_1 \ldots i_s k]}_\sigma \big) \omega^\sigma + s! \big(A^{i_1 \ldots i_s j}_\sigma - A^{[i_1 \ldots i_s j]}_\sigma \big) \omega^\sigma_j \big] \wedge ds_{i_1 \ldots i_s}$$
$$\mathbb{T} = \frac{1}{(s+1)!} \big[s! A^{[i_1 \ldots i_s i]}_\sigma \omega^\sigma \big] \wedge ds_{i_1 \ldots i_s i}.$$

On the other hand the recurrence formulae of our decomposition give as results:

$$\mathbb{E}' = \big[\big(A^{i_1 \ldots i_s}_\sigma - d_k A^{[i_1 \ldots i_s k]}_\sigma \big) \omega^\sigma + \big(A^{i_1 \ldots i_s j}_\sigma - A^{[i_1 \ldots i_s j]}_\sigma \big) \omega^\sigma_j \big] \wedge ds_{i_1 \ldots i_s}$$
$$\mathbb{T}' = \frac{1}{(s+1)} \big[A^{[i_1 \ldots i_s i]}_\sigma \omega^\sigma \big] \wedge ds_{i_1 \ldots i_s i}$$

which prove our proposition. *q.e.d.*

Proposition 4.4. *Let $r = 2$ and $s = 1$ and let $\rho \in \Omega^2_n W$ be a 1-contact $(n-1)$-horizontal form on W^2. Then there exists a variational morphism*

$$\alpha\colon J^4W \longrightarrow (J^1V(W))^* \otimes A_{n-2}(U)$$

such that the decompositions

$$< \mathbb{V}_\rho | J^2\Xi >=< \mathbb{E}|J^2\Xi > + Div(< \mathbb{T}|J^1\Xi >) =< \mathbb{E}'|J^2\Xi > + Div(< \mathbb{T}'|J^1\Xi >)$$

are related by

$$\mathbb{E}' = \mathbb{E} - \mathcal{D}\alpha\,, \quad \mathbb{T}' = \mathbb{T} + \alpha\,, \tag{21}$$

where

$$\mathcal{D}\alpha\colon J^5W \longrightarrow (J^2V(W))^* \otimes A_{n-1}(U)$$

is the unique variational morphism such that

$$< \mathcal{D}\alpha|J^2\Xi >= Div(< \alpha|J^1\Xi >)$$

Proof. Again we show this relation by direct calculations. For $r = 2$ and $s = 1$, we have locally

$$p_1\rho = \left(A^i_\sigma \omega^\sigma + A^{ij_1}_\sigma \omega^\sigma_{j_1} + A^{ij_1j_2}_\sigma \omega^\sigma_{j_1j_2}\right) \wedge ds_i \in \Omega^3_n W.$$

It is not difficult to see that, choosing the fibered connection whose coefficients vanish in the coordinate domain $W \subset Y$, we obtain the following volume part and boundary part:

$$\begin{aligned}
&< \mathbb{E}|J^2\Xi >= \Big[\Big(A^i_\sigma - d_a A^{[ia]}_\sigma - \frac{2}{3} d_b d_a A^{[ib]a}_\sigma\Big)\Xi^\sigma + \\
&+\Big(A^{(ij_1)}_\sigma + \frac{2}{3} d_a A^{aij_1}_\sigma - \frac{2}{3} d_a A^{(ij_1)a}_\sigma\Big)\Xi^\sigma_{j_1} + A^{(ij_1j_2)}_\sigma \Xi^\sigma_{j_1j_2}\Big] \wedge ds_i \\
&< \mathbb{T}|J^1\Xi >= \frac{1}{2}\Big[\Big(A^{[i_1i_2]}_\sigma - \frac{2}{3} d_a A^{[i_1i_2]a}_\sigma\Big)\Xi^\sigma + \frac{4}{3} A^{[i_1i_2]j}_\sigma \Xi^\sigma_j\Big] \wedge ds_{i_1i_2}\,.
\end{aligned} \tag{22}$$

In order to compute $\mathbb{T}'$ from equation (15), we first report

$$\chi^{ij_1} = (A^{ij_1}_\sigma - 2d_a A^{ij_1a}_\sigma)\omega^\sigma\,, \quad \chi^{ij_1j_2} = A^{ij_1j_2}_\sigma \omega^\sigma\,,$$

then

$$
\begin{aligned}
\mathbb{T}' &= \sum_{|J|=0}^{1} \frac{1}{2} d_J \chi^{[i_1 i_2]J} \wedge ds_{i_1 i_2} = \frac{1}{2}\chi^{[i_1 i_2]} \wedge ds_{i_1 i_2} + \frac{1}{2} d_a \chi^{[i_1 i_2]a} \wedge ds_{i_1 i_2} = \\
&= \frac{1}{2}(A_\sigma^{[i_1 i_2]} - 2 d_a A_\sigma^{[i_1 i_2]a})\omega^\sigma \wedge ds_{i_1 i_2} + \frac{1}{2}(d_a A_\sigma^{[i_1 i_2]a}\omega^\sigma + A_\sigma^{[i_1 i_2]a}\omega_a^\sigma) \wedge ds_{i_1 i_2} = \\
&= \big[\frac{1}{2}(A_\sigma^{[i_1 i_2]} - d_a A_\sigma^{[i_1 i_2]a})\omega^\sigma + \frac{1}{2} A_\sigma^{[i_1 i_2]a}\omega_a^\sigma\big] \wedge ds_{i_1 i_2} \,.
\end{aligned}
$$

Comparing with equation (22) it is immediate to get:

$$
\alpha := \mathbb{T}' - \mathbb{T} = \big[-\frac{1}{6} d_a A_\sigma^{[i_1 i_2]a}\omega^\sigma - \frac{1}{6} A_\sigma^{[i_1 i_2]a}\omega_a^\sigma\big] \wedge ds_{i_1 i_2}.
$$

For what concerns $\mathbb{E}'$ we have, according to the "canonical splitting"-like algorithm of Proposition 4.2:

$$
\begin{aligned}
\mathbb{E}' = \big[&(A_\sigma^i - d_a A_\sigma^{[ia]} + d_b d_a A_\sigma^{[ib]a})\omega^\sigma + (A_\sigma^{ij_1} - d_a A_\sigma^{[ia]j_1} - A_\sigma^{[ij_1]} + d_a A_\sigma^{[ij_1]a})\omega_{j_1}^\sigma + \\
&+(A_\sigma^{ij_1 j_2} - A_\sigma^{[ij_1]j_2})\omega_{j_1 j_2}^\sigma\big] \wedge ds_i.
\end{aligned}
$$

Thence the difference $\mathbb{E}' - \mathbb{E}$ is given by:

$$
\begin{aligned}
\mathbb{E}' - \mathbb{E} = \big[&\frac{1}{3} d_b d_a A_\sigma^{[ib]a}\omega^\sigma + (d_a A_\sigma^{[ai]j_1} + \\
&+ d_a A_\sigma^{[ij_1]a} + \frac{2}{3} d_a A_\sigma^{(ij_1)a} - \frac{2}{3} d_a A_\sigma^{aij_1})\omega_{j_1}^\sigma + (A_\sigma^{(ij_1)j_2} - A_\sigma^{(ij_1 j_2)})\omega_{j_1 j_2}^\sigma\big] \wedge ds_i \,.
\end{aligned} \tag{23}
$$

On the other hand

$$
\begin{aligned}
&-\mathcal{D}\alpha = \mathcal{D}(-\alpha) = \\
&= \big[\frac{1}{3} d_b d_a A_\sigma^{[ib]a}\omega^\sigma + (\frac{1}{3} d_a A_\sigma^{[ij_1]a} + \frac{1}{3} d_a A_\sigma^{[ia]j_1})\omega_{j_1}^\sigma + \frac{1}{3} A_\sigma^{[ij_1]j_2}\omega_{j_1 j_2}^\sigma\big] \wedge ds_i \,.
\end{aligned} \tag{24}
$$

To complete our discussion we just have to compare the coefficients of the respective contact forms in equations (23) and (24). The coefficients of ω^σ are precisely the same. Let us take the coefficients of $\omega_{j_1}^\sigma$ and compute their difference:

$$
\begin{aligned}
&\frac{2}{3}(d_a A_\sigma^{[ij_1]a} + 2 d_a A_\sigma^{[ai]j_1} + d_a A_\sigma^{(ij_1)a} - d_a A_\sigma^{aij_1}) = \\
&= \frac{2}{3}(d_a A_\sigma^{ij_1 a} + d_a A_\sigma^{aij_1} - d_a A_\sigma^{iaj_1} - d_a A_\sigma^{aij_1}) = 0.
\end{aligned}
$$

Finally, consider the coefficient of $\omega^{\sigma}_{j_1 j_2}$ and recast it as follows:

$$
\begin{aligned}
&A_{\sigma}^{(ij_1)j_2} - A_{\sigma}^{(ij_1j_2)} = \frac{1}{2}(A_{\sigma}^{ij_1j_2} + A_{\sigma}^{j_1ij_2}) - \frac{1}{3}(A_{\sigma}^{ij_1j_2} + A_{\sigma}^{j_1ij_2} + A_{\sigma}^{j_2j_1i}) = \\
&= \frac{1}{6}A_{\sigma}^{ij_1j_2} + \frac{1}{6}A_{\sigma}^{j_1ij_2} - \frac{1}{3}A_{\sigma}^{j_2j_1i} = \frac{1}{6}A_{\sigma}^{ij_1j_2} - \frac{1}{6}A_{\sigma}^{j_1ij_2} + \frac{1}{3}A_{\sigma}^{j_1j_2i} - \frac{1}{3}A_{\sigma}^{j_2j_1i} = \\
&= \frac{1}{3}A_{\sigma}^{[ij_1]j_2} + \frac{2}{3}A_{\sigma}^{[j_1j_2]i}.
\end{aligned}
$$

When the last right hand side of the equations above is contracted with $\omega^{\sigma}_{j_1 j_2}$ (which is symmetric in $(j_1 j_2)$) only the first term $\frac{1}{3}A_{\sigma}^{[ij_1]j_2}$ remains, which is exactly the coefficient in equation (24), as we wanted to show. *q.e.d.*

If we consider the case of $\mathbb{E}$ given by the momentum morphism according to the formulation of [3] and [4], then the relation (21) provides another *Lepage equivalent* than the usual Poincaré-Cartan form (see also the discussion about this point in [18]). The uniqueness and globality properties of the terms $\mathbb{E}'$ and $\mathbb{T}'$ remain to be deeper investigated; in particular, when defined globally, the difference $\mathbb{E}' - \mathbb{E}$ is a closed form, thus defining a de Rham cohomology class. This feature deals with other topological aspects of Lagrangian field theories [2, 5, 6, 17, 19] which will be the subject of future research.

5 The Krupka–Betounes equivalent for first order theories and a first glance to the second order

Suppose we have a fibered manifold $\pi\colon Y \longrightarrow X$, with $\dim X = n$ and $\dim Y = n + m$. Consider a local chart (x^i, y^σ), $1 \le i \le n$, $1 \le \sigma \le m$ on Y, then on J^1Y we have local fibered coordinates $(x^i, y^\sigma, y^\sigma_j)$, with $1 \le i, j \le n$ and $1 \le \sigma \le m$.

Let us consider a Lagrangian λ, defined on J^1Y, locally expressed as $\lambda = \mathscr{L}(x^i, y^\sigma, y^\sigma_j)ds$. As we already remarked, in [18] a Residual operator for lower degree forms was obtained by the first author for the antisymmetric case; see Remark 4.1 above. Olga Rossi [20] conjectured that, making use of it, the Krupka–Betounes Lepage equivalent for first order theories could be obtained by the application of the following recurrence formulae:

$$\begin{aligned}
&\rho_1 = \lambda - p_1\mathcal{R}(d\lambda) = \theta_\lambda \quad \text{(Poincaré-Cartan form of the Lagrangian)}\\
&\rho_2 = \rho_1 - p_2\mathscr{R}(d\rho_1)\\
&\rho_3 = \rho_2 - p_3\mathscr{R}(d\rho_2)\\
&\dots\\
&\rho_n = \rho_{n-1} - p_n\mathscr{R}(d\rho_{n-1})
\end{aligned}$$

with $\mathcal{R}$ and $\mathscr{R}$ the residual operator for top forms and lower degree forms, respectively.

In the following, in view of a generalization to the second order case, we apply the recurrence formulae by using the characterization of the Residual operator for lower degree forms as given in the present paper by Proposition 4.1.

We compute explicitly the forms $\rho_1, \dots, \rho_n$. First we have the well-known Poincaré-Cartan form of λ:

$$\rho_1 = \theta_\lambda = \mathscr{L}ds + p^i_\sigma \omega^\sigma \wedge ds_i \qquad p^i_\sigma := \frac{\partial \mathscr{L}}{\partial y^\sigma_i}.$$

In order to compute ρ_2 we observe that the Residual operator $\mathscr{R}$ does not change the order of contactness of its argument, i.e. $p_2\mathscr{R}(d\rho_1) = \mathscr{R}(p_2 d\rho_1)$. Thus we can reduce ourselves to consider only the 2-contact component of the differential of ρ_1. In particular:

$$\begin{aligned}
&p_2 d\rho_1 = \omega^\sigma \wedge (\partial_\sigma p^i_\nu \omega^\nu \wedge ds_i) + \omega^\sigma_j \wedge (\partial^j_\sigma p^i_\nu \omega^\nu \wedge ds_i) =\\
&= \omega^\sigma \wedge [\partial_\sigma p^i_\nu \omega^\nu \wedge ds_i - d_j(\partial^j_\sigma p^i_\nu \omega^\nu \wedge ds_i)] + d_j[\omega^\sigma \wedge (\partial^j_\sigma p^i_\nu \omega^\nu \wedge ds_i)]\,.
\end{aligned}$$

Specializing Proposition 4.1 we have:

$$\mathscr{R}(p_2 d\rho_1) = \frac{1}{2}\partial^j_\sigma p^i_\nu \omega^\sigma \wedge \omega^\nu \wedge ds_{ij}.$$

Thus

$$\rho_2 = \rho_1 - p_2\mathscr{R}(d\rho_1) = \mathscr{L}ds + p^i_\sigma \omega^\sigma \wedge ds_i + \frac{1}{2}\partial^i_\sigma p^j_\nu \omega^\sigma \wedge \omega^\nu \wedge ds_{ij}.$$

Now we compute $\rho_3 = \rho_2 - p_3\mathscr{R}(d\rho_2)$. Again we can restrict our attention to the term

$$\begin{aligned}
&p_3 d\rho_2 = \frac{1}{2}(\partial_{\sigma_1}\partial^i_{\sigma_2} p^j_{\sigma_3} \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \omega^{\sigma_3} \wedge ds_{ij} + \partial^k_{\sigma_1}\partial^i_{\sigma_2} p^j_{\sigma_3} \omega^{\sigma_1}_k \wedge \omega^{\sigma_2} \wedge \omega^{\sigma_3} \wedge ds_{ij})\\
&= \omega^{\sigma_1} \wedge (\dots) + d_k[\omega^{\sigma_1} \wedge \frac{1}{2}(\partial^k_{\sigma_1}\partial^i_{\sigma_2} p^j_{\sigma_3} \omega^{\sigma_2} \wedge \omega^{\sigma_3} \wedge ds_{ij})].
\end{aligned}$$

Thence

$$p_3\mathscr{R}(d\rho_2) = -\frac{1}{6}\partial^k_{\sigma_1}\partial^i_{\sigma_2}p^j_{\sigma_3}\omega^{\sigma_1}\wedge\omega^{\sigma_2}\wedge\omega^{\sigma_3}\wedge ds_{ijk}$$

and finally

$$\rho_3 = \rho_2 - p_3\mathscr{R}(d\rho_2) = \mathscr{L}ds + p^i_\sigma\omega^\sigma\wedge ds_i + \frac{1}{2!}\partial^{i_1}_{\sigma_1}p^{i_2}_{\sigma_2}\omega^{\sigma_1}\wedge\omega^{\sigma_2}\wedge ds_{i_1i_2} +$$
$$+\frac{1}{3!}\partial^{i_1}_{\sigma_1}\partial^{i_2}_{\sigma_2}p^{i_3}_{\sigma_3}\omega^{\sigma_1}\wedge\omega^{\sigma_2}\wedge\omega^{\sigma_3}\wedge ds_{i_1i_2i_3}.$$

Proceeding this way it is straightforward to see that at the end we obtain the following result:

$$\rho_n = \mathscr{L}ds + \sum_{q=1}^{n}\frac{1}{q!}\frac{\partial^q\mathscr{L}}{\partial y^{\sigma_1}_{i_1}\dots\partial y^{\sigma_q}_{i_q}}\omega^{\sigma_1}\wedge\dots\wedge\omega^{\sigma_q}\wedge ds_{i_1\dots i_q} \tag{25}$$

which is known as the *Krupka-Betounes equivalent* of the Lagrangian λ [1, 10, 11, 21, 23].

5.1 Lepage equivalents for second order theories

Let us now consider a second order Lagrangian $\lambda = \mathscr{L}(x^i, y^\sigma, y^\sigma_j, y^\sigma_{jk})ds$. As a first step we compute the Poincaré-Cartan form of the Lagrangian using the first of Olga Rossi's recurrence formulae: $\rho_1 = \lambda - p_1\mathcal{R}(d\lambda) = \theta_\lambda$.

$$d\lambda = p_\sigma\omega^\sigma\wedge ds + p^j_\sigma\omega^\sigma_j\wedge ds + p^{jk}_\sigma\omega^\sigma_{jk}\wedge ds =$$
$$= d_jd_k(p^{jk}_\sigma\omega^\sigma\wedge ds) + d_j[(p^j_\sigma - 2d_kp^{jk}_\sigma)\omega^\sigma\wedge ds] + (d_kd_jp^{jk}_\sigma - d_jp^j_\sigma + p_\sigma)\omega^\sigma\wedge ds.$$

Using the formula for the residual operator in this particular case:

$$\mathcal{R} = \sum_{|I|=0}^{1}(-1)^1 d_I\chi^{Ij}\wedge ds_j,$$

we obtain:

$$p_1\mathcal{R}(d\lambda) = -[(p^j_\sigma - 2d_kp^{jk}_\sigma + d_kp^{jk}_\sigma)\omega^\sigma\wedge ds_j + p^{jk}_\sigma\omega^\sigma_k\wedge ds_j].$$

Then, as expected, the Poincaré-Cartan form of the Lagrangian λ is:

$$\theta_\lambda = \mathscr{L}ds + (p^j_\sigma - d_kp^{jk}_\sigma)\omega^\sigma\wedge ds_j + p^{jk}_\sigma\omega^\sigma_k\wedge ds_j.$$

We now rename $f^j_\sigma := p^j_\sigma - d_k p^{jk}_\sigma$ and $f^{jk}_\sigma := p^{jk}_\sigma$ and continue computing the second of Rossi's recurrence formulae: $\rho_2 = \theta_\lambda - p_2\mathscr{R}(d\theta_\lambda)$.

We are interested only in the 2-contact component of $d\theta_\lambda$ because $p_2\mathscr{R} = p_2\mathscr{R}p_2$, *i.e.* the Residual operator $\mathscr{R}$ does not increase the order of contactness of its argument, thus let us write

$$\begin{aligned} p_2(d\theta_\lambda) &= \omega^\sigma \wedge (\partial_\sigma f^i_{\sigma_2}\omega^{\sigma_2} \wedge ds_i - \partial^{j_1}_{\sigma_1} f^i_\sigma \omega^{\sigma_1}_{j_1} \wedge ds_i - \partial^{j_1 j_2}_{\sigma_1} f^i_\sigma \omega^{\sigma_1}_{j_1 j_2} \wedge ds_i + \\ &+\partial_\sigma f^{ij_2}_{\sigma_2}\omega^{\sigma_2}_{j_2} \wedge ds_i) + \omega^\sigma_j \wedge (\partial^j_\sigma f^{ij_2}_{\sigma_2}\omega^{\sigma_2}_{j_2} \wedge ds_i - \partial^{j_1 j_3}_{\sigma_1} f^{ij}_\sigma \omega^{\sigma_1}_{j_1 j_3} \wedge ds_i)\,. \end{aligned}$$

Now we recast this expression in the form $\sum_{|I|=0}^1 d_I(\omega^\sigma \wedge \xi^I_\sigma)$ and apply the integration by parts lemma, so to get for the Residual operator

$$\begin{aligned} p_2\mathscr{R}(d\theta_\lambda) &= \frac{1}{2}\partial^j_\sigma f^{ij_2}_{\sigma_2}\omega^{\sigma_2}_{j_2} \wedge ds_{ij} - \frac{1}{2}\partial^{j_1 j_3}_{\sigma_1} f^{ij}_\sigma \omega^{\sigma_1}_{j_1 j_3} \wedge ds_{ij} = \\ &= \frac{1}{2}\partial^j_\sigma f^{ij_2}_{\sigma_2}\omega^{\sigma_2}_{j_2} \wedge ds_{ij}\,, \end{aligned}$$

being f^{ij}_σ symmetric in (ij). Hence we obtain the following formula for ρ_2:

$$\rho_2 = \mathscr{L}ds + f^i_\sigma\omega^\sigma \wedge ds_i + f^{ij}_\sigma\omega^\sigma_j \wedge ds_i + \frac{1}{2}\partial^{i_1}_{\sigma_1} f^{i_2 j_2}_{\sigma_2}\omega^{\sigma_1} \wedge \omega^{\sigma_2}_{j_2} \wedge ds_{i_1 i_2}$$

It turns out that it is possible to apply this reasoning at any step of Rossi's recurrence formulae, *e.g.* at the third step we get

$$p_3\mathscr{R}(d\rho_2) = -\frac{1}{6}\partial^{i_1}_{\sigma_1}\partial^{i_2}_{\sigma_2} f^{i_3 j}_\sigma \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \omega^\sigma_j \wedge ds_{i_1 i_2 i_3}.$$

Proceeding this way, the final expression of ρ_n has the form:

$$\begin{aligned} \rho_n &= \mathscr{L}ds + f^i_\sigma\omega^\sigma \wedge ds_i + \\ &+\sum_{q=1}^n \frac{1}{q!}\frac{\partial\mathscr{L}}{\partial y^{\sigma_1}_{i_1}\dots\partial y^{\sigma_{q-1}}_{i_{q-1}}\partial y^{\sigma_q}_{i_q j}}\,\omega^{\sigma_1} \wedge \dots \wedge \omega^{\sigma_{q-1}} \wedge \omega^{\sigma_q}_j \wedge ds_{i_1\dots i_q} \end{aligned}$$

We remark that the integration by parts is not uniquely defined in the case $r = 2$. Indeed, if we take into account in the decomposition of $p_2(d\rho_1) = p_2(d\theta_\lambda)$ also the term $\omega^\sigma_j \wedge (\partial^j_\sigma f^i_{\sigma_2}\omega^{\sigma_2} \wedge ds_i)$ and work on it as above, we get an additional term in ρ_n, thus obtaining a sort of generalization of the Krupka-Betounes equivalent at the second order, namely :

$$\begin{aligned} \rho_n &= \mathscr{L}ds + \sum_{q=1}^n \frac{1}{q!}\frac{\partial\mathscr{L}}{\partial y^{\sigma_1}_{i_1}\dots\partial y^{\sigma_{q-1}}_{i_{q-1}}\partial y^{\sigma_q}_{i_q j}}\,\omega^{\sigma_1} \wedge \dots \wedge \omega^{\sigma_{q-1}} \wedge \omega^{\sigma_q}_j \wedge ds_{i_1\dots i_q} + \\ &+f^i_\sigma\omega^\sigma \wedge ds_i + \sum_{q=1}^{n-1}\frac{1}{(q+1)!}\frac{\partial f^{i_{q+1}}_{\sigma_{q+1}}}{\partial y^{\sigma_1}_{i_1}\dots\partial y^{\sigma_q}_{i_q}}\,\omega^{\sigma_1} \wedge \dots \wedge \omega^{\sigma_q} \wedge \omega^{\sigma_{q+1}} \wedge ds_{i_1\dots i_q i_{q+1}}\,, \end{aligned}$$

which exactly reduces to the Krupka-Betounes equivalent (25), when the Lagrangian is of order $r = 1$ (see [16] for a review on Lepage equivalents of order $r \geq 1$).

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