The SU(2) Lie-Poisson Algebra and its Descendants

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In this paper, a novel discrete algebra is presented which follows by combining the SU(2) Lie-Poisson bracket with the discrete Frenet equation. Physically, the construction describes a discrete piecewise linear string in $\mathbb{R}^3$. The starting point of our derivation is the discrete Frenet frame assigned at each vertix of the string. Then the link vector that connect the neighbouring vertices assigns the SU(2) Lie-Poisson bracket. Moreover, the same bracket defines the transfer matrices of the discrete Frenet equation which relates two neighbouring frames along the string. The procedure extends in a self-similar manner to an infinite hierarchy of Poisson structures. As an example, the first descendant of the SU(2) Lie-Poisson structure is presented in detail. For this, the spinor representation of the discrete Frenet equation is employed, as it converts the brackets into a computationally more manageable form. The final result is a nonlinear, nontrivial and novel Poisson structure that engages four neighbouring vertices.

I. INTRODUCTION

The SU(2) Lie-Poisson bracket is an example of a Poisson structure that is also a Lie algebra. The bracket was originally introduced by Lie [1]. A systematic investigation started much later, with seminal contributions in particular by Lichnerowicz [2], Kirillov [3] and Weinstein [4]. Subsequently Lie-Poisson structures [5] have been investigated widely, with a large number of fundamental physics applications from string theory and integrable systems to conformal and topological field theories [6, 7].

This paper proves that Poisson structures can also be relevant in connection of effective theory descriptions of discrete stringlike objects. Discrete piecewise linear strings have appeared from models of proteins in terms of the Cα backbone [8] to considerations of segmented string evolution in de Sitter and anti-de Sitter spaces [9]. In addition, they have important applications in robotics and 3D virtual reality [10].

The paper is arranged as follows. Initially, the descendants of the SU(2) Lie-Poisson structure that relates to the structure of a discrete piecewise linear polygonal string are considered. In addition, the model space and its reduction in the case of the standard SU(2) Lie-Poisson bracket is reviewed. Then the formalism of the discrete Frenet frames [11] and its self-similar hierarchical structure is presented. Finally, following the results of [12], the self-similar structure is converted into a spinor representation, while the Poisson brackets in terms of the SU(2) Lie-Poisson structure are introduced. That way, an infinite hierarchy of Poisson structures can be assigned to piecewise linear string as descendants of the canonical SU(2) Lie-Poisson structure. To conclude, an explicit construction of the first level descendant in this hierarchy is presented in detail.

II. THE MODEL SPACE AND THE LIE-POISSON STRUCTURE

This preparatory section summarises known results on the model space of SU(2) representations and the SU(2) Lie-Poisson structure. The starting point is a four dimensional phase space $\mathbb{R}^4$ equipped with a canonical symplectic structure and Darboux coordinates (q_1, p_1, q_2, p_2)

$$\{p^\alpha, q^\beta\} = -\delta^{\alpha\beta},$$

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combined into two complex ones

$$w^\alpha = \frac{1}{\sqrt{2}}(p^\alpha + iq^\alpha), \qquad (\alpha = 1,2). \tag{1}$$

Their norm is set to be ρ, ie.

$$||w^1||^2 + ||w^2||^2 = 2\rho, \tag{2}$$

while the associated Poisson brackets have the simple form,

$$\{w^\alpha, \bar{w}^\beta\} = i\delta^{\alpha\beta}, \qquad \{w^\alpha, w^\beta\} = \{\bar{w}^\alpha, \bar{w}^\beta\} \;=\; 0. \tag{3}$$

Next define the three component unit length vector

$$t^a \;=\; -\frac{1}{2\rho}\begin{pmatrix}\bar{w}^1 & \bar{w}^2\end{pmatrix}\sigma^a\begin{pmatrix}w^1\\ w^2\end{pmatrix}, \qquad (a = 1,2,3), \tag{4}$$

where σ^a are the Pauli matrices. Then, the t^a components obey the SU(2) Lie-Poisson bracket

$$\{t^a, t^b\} = \frac{1}{\rho}\,\epsilon^{abc}t^c, \tag{5}$$

associated with the identity

$$\{t^a, \rho\} = 0. \tag{6}$$

Therefore, ρ is a Casimir element while the phase space (1) is a model space of SU(2) representations. Note that, different values of ρ correspond to different representations. The bracket (5) determines a Poisson structure since:

It is antisymmetric, ie., any two functions A and B satisfy

$$\{A, B\} = -\{B, A\}. \tag{7}$$

It obeys both the Jacobi identity

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0 \tag{8}$$

and the Leibnitz rule

$$\{A, BC\} \;=\; \{A, B\}C + B\{A, C\}. \tag{9}$$

Note that the Jacobi identity coincides with the Schouten bracket of the Poisson bi-vector field

$$\Lambda \;=\; \epsilon^{abc}t^c\partial_a \wedge \partial_b, \tag{10}$$

from which the Leibnitz rule follows directly.

Since the rank of the antisymmetric matrix $\epsilon^{abc}t^c$ is two, the bracket in (5) does not determine a symplectic structure. However, the Poisson bracket (3) is symplectic with the closed and non-degenerate two-form

$$\omega \;=\; dp_1 \wedge dq_1 + dp_2 \wedge dq_2 \;=\; idw_1 \wedge dw_1^\star + idw_2 \wedge dw_2^\star. \tag{11}$$

Therefore, a Darboux coordinate representation of (5) can be derived by introducing the harmonic coordinates

$$\begin{pmatrix}w^1\\ w^2\end{pmatrix} \;=\; \sqrt{2\rho}\begin{pmatrix}\cos\frac{\theta}{2}\, e^{i(\varphi+\phi)/2}\\ \sin\frac{\theta}{2}\, e^{i(\varphi-\phi)/2}\end{pmatrix}, \tag{12}$$

and thus, the unit length vector (4) simplifies to

$$\mathbf{t} = \begin{pmatrix}t^1\\ t^2\\ t^3\end{pmatrix} \;=\; \begin{pmatrix}\cos\phi\sin\theta\\ \sin\phi\sin\theta\\ \cos\theta\end{pmatrix}. \tag{13}$$

These coordinates foliate $\mathbb{R}^4 \sim \mathbb{R}^1 \times \mathbb{S}^3 \sim \mathbb{R}^1 \times \mathbb{S}^1 \times \mathbb{S}^2$ where (φ, ϕ, θ) are the angular coordinates and $\sqrt{2\rho}$ the radii. That way, the symplectic two-form (11) becomes

$$\omega \;=\; d\rho \wedge d\varphi + \cos\theta d\rho \wedge d\phi + \rho\, d\cos\theta \wedge d\phi \;\equiv\; d\rho \wedge d\varphi + d(\rho\cos\theta) \wedge d\phi, \tag{14}$$

with the only non-vanishing Poisson brackets given by

$$\{\rho, \varphi\} \;=\; -1, \qquad\qquad \{\rho\cos\theta, \phi\} \;=\; -1. \tag{15}$$

Finally, by setting

$$\chi \;=\; \epsilon\varphi, \tag{16}$$

and taking the Inönü-Wigner contraction limit ($\epsilon \to 0$) of the system (15), only the second bracket survives. The latter corresponds to the symplectic Poisson bracket on $\mathbb{S}^2$ together with its closed two-form (unique up to coordinate changes), that coincides with the last term in (14). Note that the coordinate ρ appears only as a Casimir element of the Lie-Poisson bracket. Thus, for simplicity, in what follows $\rho = 1$.

III. DISCRETE FRENET EQUATION AND SELF-SIMILARITY

A. Vector representation of the discrete Frenet frames

In this section descendants of the SU(2) Lie-Poisson bracket defined by (5), that arise in connection of open and piecewise linear polygonal strings $\mathbf{x}(s) \in \mathbb{R}^3$, are constructed. To set the stage, let s be the arc length parameter with values $s \in [0, L]$ while L is the length of the string. Also, $\mathcal{V}_i$ with $i = 0, ..., n$ are the vertices that characterise the string located at the points $\mathbf{x}(s_i) = \mathbf{x}_i$. Then, neighbouring vertices are connected by the line segments

$$\mathbf{x}(s) \;=\; \frac{s - s_i}{s_{i+1} - s_i}\,\mathbf{x}_{i+1} \;-\; \frac{s - s_{i+1}}{s_{i+1} - s_i}\,\mathbf{x}_i, \qquad\qquad s \in (s_i, s_{i+1}),$$

and are separated by the distances

$$|\mathbf{x}_{i+1} - \mathbf{x}_i| = s_{i+1} - s_i \;\equiv\; \Delta_i.$$

The discrete Frenet frames are defined by the orthogonal triplets $(\mathbf{t}, \mathbf{n}, \mathbf{b})_i$ at the vertices $\mathcal{V}_i$ as follows: The unit length tangent vectors $\mathbf{t}_i$ point from $\mathcal{V}_i$ to $\mathcal{V}_{i+1}$

$$\mathbf{t}_i \;=\; \frac{1}{\Delta_i}(\mathbf{x}_{i+1} - \mathbf{x}_i), \tag{17}$$

the unit length binormal vectors are

$$\mathbf{b}_i = \frac{\mathbf{t}_{i-1} \times \mathbf{t}_i}{|\mathbf{t}_{i-1} \times \mathbf{t}_i|}, \tag{18}$$

and the unit length normal vectors $\mathbf{n}_i$ are computed from

$$\mathbf{n}_i \;=\; \mathbf{b}_i \times \mathbf{t}_i \;=\; \frac{-\mathbf{t}_{i-1} + (\mathbf{t}_{i-1} \cdot \mathbf{t}_i)\,\mathbf{t}_i}{|\mathbf{t}_{i-1} + (\mathbf{t}_{i-1} \cdot \mathbf{t}_i)\,\mathbf{t}_i|}. \tag{19}$$

In addition, the transfer matrix $\mathcal{R}_{i+1,i}$ maps the discrete Frenet frames between the neighbouring vertices $\mathcal{V}_i$ and $\mathcal{V}_{i+1}$

$$\begin{pmatrix} \mathbf{n} \\ \mathbf{b} \\ \mathbf{t} \end{pmatrix}_{i+1} = \mathcal{R}_{i+1,i} \begin{pmatrix} \mathbf{n} \\ \mathbf{b} \\ \mathbf{t} \end{pmatrix}_i = \begin{pmatrix} \cos\tau\cos\kappa & \sin\tau\cos\kappa & -\sin\kappa \\ -\sin\tau & \cos\tau & 0 \\ \cos\tau\sin\kappa & \sin\tau\sin\kappa & \cos\kappa \end{pmatrix}_i \begin{pmatrix} \mathbf{n} \\ \mathbf{b} \\ \mathbf{t} \end{pmatrix}_i. \tag{20}$$

Here κ_{i+1} is the bond angle and τ_{i+1} is the torsion angle. [Note that, the transfer matrix $\mathcal{R}_{i+1,i}$ $\in$SO(3) engages only two of the Euler angles $(\kappa, \tau)_i$ since the third Euler angle becomes removed by the orthogonality of $\mathbf{b}_i$ and $\mathbf{t}_{i-1}$.]

The torsion and bond angles (κ_i, τ_i) are expressible in terms of the tangent vectors only. This observation follows directly from equation (20) since

$$\cos\kappa_i \;=\; \mathbf{t}_{i+1}\cdot\mathbf{t}_i, \tag{21}$$

while

$$\cos\tau_i \;=\; \mathbf{b}_{i+1}\cdot\mathbf{b}_i \;=\; \frac{\mathbf{t}_i\times\mathbf{t}_{i+1}}{|\mathbf{t}_i\times\mathbf{t}_{i+1}|}\cdot\frac{\mathbf{t}_{i-1}\times\mathbf{t}_i}{|\mathbf{t}_{i-1}\times\mathbf{t}_i|}. \tag{22}$$

In addition, the bond angle engages three vertices while the torsion angle engages four vertices along the string.

The aforementioned construction can be extended into an infinite hierarchy (for an infinite length string) in a self-similar manner. To do so the transfer matrix (20) is used to introduce a 2^{nd} level orthonormal triplet of vectors $(\mathbf{T},\mathbf{N},\mathbf{B})_i$. The components of the vector $\mathbf{T}_i$ are defined in terms of the last row of (20)

$$\mathbf{T}_i \;=\; \begin{pmatrix}\cos\tau_i\sin\kappa_i\\ \sin\tau_i\sin\kappa_i\\ \cos\kappa_i\end{pmatrix}, \tag{23}$$

while the corresponding 2^{nd} level binormal and normal vectors, in analogy with (18) and (19), are defined as

$$\mathbf{B}_i \;=\; \frac{\mathbf{T}_{i-1}\times\mathbf{T}_i}{|\mathbf{T}_{i-1}\times\mathbf{T}_i|}, \qquad\qquad \mathbf{N}_i \;=\; \frac{-\mathbf{T}_{i-1}+(\mathbf{T}_{i-1}\cdot\mathbf{T}_i)\,\mathbf{T}_i}{|\mathbf{T}_{i-1}+(\mathbf{T}_{i-1}\cdot\mathbf{T}_i)\,\mathbf{T}_i|}. \tag{24}$$

Then the corresponding equation (20) determines the 2^{nd}-level transfer matrix

$$\begin{pmatrix}\mathbf{N}\\ \mathbf{B}\\ \mathbf{T}\end{pmatrix}_{i+1} \;=\; \mathcal{R}_{i+1,i}\begin{pmatrix}\mathbf{N}\\ \mathbf{B}\\ \mathbf{T}\end{pmatrix}_i \;\equiv\; \begin{pmatrix}\cos\mathcal{T}\cos\mathcal{K} & \sin\mathcal{T}\cos\mathcal{K} & -\sin\mathcal{K}\\ -\sin\mathcal{T} & \cos\mathcal{T} & 0\\ \cos\mathcal{T}\sin\mathcal{K} & \sin\mathcal{T}\sin\mathcal{K} & \cos\mathcal{K}\end{pmatrix}_i\begin{pmatrix}\mathbf{N}\\ \mathbf{B}\\ \mathbf{T}\end{pmatrix}_i. \tag{25}$$

with $(\mathcal{K},\mathcal{T})_i$ the 2^{nd}-level bond and torsion angles evaluated in terms of the 2^{nd}-level $\mathbf{T}_i$ in analogy to equations (21) and (22).

The construction can be extended to the next level. That is, using the last row of (25) the formulation (23) is used to introduce the 3^{rd}-level tangent vectors. From these, the 3^{rd} level vectors (24) and transfer matrix (25) are obtained. The construction can then be continued to higher levels (in a self-similar manner) and thus, an infinite hierarchy is obtained. In particular, every vector and angle that appears in this self-similar hierarchy, can be expressed recursively in terms of the initial tangent vectors $\mathbf{t}_i$.

B. Spinor representation of the discrete Frenet equation

In this section the spinorial form of the discrete Frenet equation (20) is presented. To do so, a two component spinor is assigned to each link that connects the vertices $\mathcal{V}_i$ and $\mathcal{V}_{i+1}$, that is,

$$\psi_i = \begin{pmatrix}z_1\\ z_2\end{pmatrix}^i. \tag{26}$$

The z_α^i (for $\alpha = 1,2$) are complex variables assigned to the link. Then, the unit length tangent vectors $\mathbf{t}_i$ can be expressed in terms of the spinors from a relation akin that in (4)

$$\psi_i^\dagger\hat{\sigma}\psi_i \;=\; \sqrt{g_i}\,\mathbf{t}_i, \tag{27}$$

where $\hat{\sigma} = (\sigma^1,\sigma^2,\sigma^3)$ are the Pauli matrices, $\mathbf{t}_i$ is the discrete tangent vector (17) and $\sqrt{g_i}$ is the scale factor,

$$\sqrt{g_i} \equiv \left(|z_1|^2+|z_2|^2\right)^i. \tag{28}$$

The difference to equation (2) should be noted. From the definition (27) and using (26) one can easily derive that

$$\begin{aligned} z_1^i \;&=\; \sqrt{\frac{g_i}{2}}\left[\sqrt{t_1-it_2}\left(\frac{1+t_3}{1-t_3}\right)^{1/4}\right]^i,\\ z_2^i \;&=\; \sqrt{\frac{g_i}{2}}\left[\sqrt{t_1+it_2}\left(\frac{1-t_3}{1+t_3}\right)^{1/4}\right]^i, \end{aligned} \tag{29}$$

while in terms of the local coordinates (13) one obtains

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^i \;=\; \sqrt{g_i}\begin{pmatrix} \cos\frac{\theta}{2}\, e^{i\phi/2} \\ \sin\frac{\theta}{2}\, e^{-i\phi/2} \end{pmatrix}^i . \tag{30}$$

In analogy to (12) the value of the overall factor $\sqrt{g_i}$ can be changed and let us (for simplicity) set $g_i = 1$.

Next the conjugation operation $\mathcal{C}$ is introduced to create the conjugate spinor $\bar{\psi}_i$,

$$\mathcal{C}\,\psi_i \;=\; -i\sigma_2\psi_i^\star \;=\; \begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 \end{pmatrix}^i \;\equiv\; \bar{\psi}_i, \tag{31}$$

so that

$$\psi_i^\dagger \bar{\psi}_i \;=\; 0.$$

Together the two spinors ψ_i and $\bar{\psi}_i$ define the 2×2 matrix

$$\mathfrak{u}_i = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}^i , \tag{32}$$

where

$$\psi_i = \mathfrak{u}_i \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \bar{\psi}_i = \mathfrak{u}_i \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Finally, to derive the spinorial discrete Fernet equation in a matrix form, a Majorana spinor is constructed from the two spinors (26) and (31) by setting

$$\Psi_i = \begin{pmatrix} -\bar{\psi} \\ \psi \end{pmatrix}^i ,$$

one can now introduce a spinorial transfer matrix $\mathcal{U}_{i+1,i}$ that relates the Majorana spinors at the neighbouring links as

$$\Psi_{i+1} \;=\; \mathcal{U}_{i+1}^\dagger \; \Psi_i. \tag{33}$$

Equation (33) is the so-called spinorial discrete Frenet equation. In analogy to (32) the matrix $\mathcal{U}_{i+1,i}$ can be expressed in terms of the vertex variables Z_a^i (for $a = 1, 2$):

$$\mathcal{U}_i \;=\; \begin{pmatrix} Z_1 & -\bar{Z}_2 \\ Z_2 & \bar{Z}_1 \end{pmatrix}^i . \tag{34}$$

The link $(z_1, z_2)^i$ and the vertex $(Z_1, Z_2)^i$ variables are connected through the discrete Frenet equation (33). In particular,

$$\begin{aligned} Z_1^{i+1} &= \bar{z}_1^i\, z_1^{i+1} + \bar{z}_2^i\, z_2^{i+1} \\ Z_2^{i+1} &= z_1^i\, z_2^{i+1} - z_2^i\, z_1^{i+1}. \end{aligned} \tag{35}$$

and the choice $\sqrt{g_i} = 1$ in (28) gives $(|Z_1|^2 + |Z_2|^2)^i = 1$.

In analogy with (25), one can introduce a 2^{nd} level spinor variables, with the ensuing 2^{nd} level spinorial Frenet equation. The construction can be repeated to higher levels, in a self-similar manner, to obtain an infinite hierarchy of spinorial discrete Frenet equations. Notably, all quantities that appear in this hierarchy can be written in terms of the complex variables (29), recursively.

IV. DESCENDANTS OF THE SU(2) LIE-POISSON BRACKET

In the case of the discrete Frenet frames, the entire self-similar hierarchy can be constructed recursively in terms of the initial tangent vectors (17). As a consequence, one can also introduce Poisson structures at all levels of the hierarchy; recall that the SU(2) Lie-Poisson brackets (5) imposed on the tangent vectors (17) take the simple form

$$\{t_i^a, t_j^b\} = \frac{1}{\Delta_i}\delta_{ij}\epsilon^{abc}t_i^c, \tag{36}$$

where Δ_i are identified as Casimir elements and for convenience the value $\Delta_i = 1$ is chosen.

Equivalently, the spinor realisation of the hierarchy can be expressed recursively in terms of the complex link variables (26). Indeed, from (36) it is straightforward to show that the link variables (29) satisfy the following algebra

$$\begin{aligned}
\{z^i_\alpha, \bar{z}^j_\alpha\} &= \frac{i}{4}\,\delta_{ij}, \qquad \alpha = 1, 2, \\
\{z^i_1, z^j_2\} &= -\frac{i}{8}\left(\frac{|z_1|^2 - |z_2|^2}{\sqrt{|z_1|^2|z_2|^2}}\right)^i \delta_{ij}, \\
\{z^i_1, \bar{z}^j_2\} &= -\frac{i}{8}\frac{1}{(\bar{z}_1 z_2)^i}\,\delta_{ij}.
\end{aligned} \tag{37}$$

While it is clear that the Poisson brackets of all the quantities that appear in the self-similar hierarchy can be evaluated recursively in terms of (36) it is not obvious that the Poisson brackets of all the components of $\mathbf{T}_i$ that appear at a given higher level of the hierarchy, form a closed algebra. If this is the case, a method is obtained to systematically generate new Poisson structures, as higher level descendants of the original SU(2) Lie-Poisson structure. In what follows, starting from the spinor representation (37) of the SU(2) Lie-Poisson bracket it is demonstrated by an explicit computation that this is the case. To do so, the Poisson brackets of the vertex variables (35) are evaluated. In particular, they are employed as coordinates to define a Poisson structure in terms of the pertinent Poisson bi-vector, that is,

$$\Lambda(Z, \bar{Z}) \;=\; \Omega^{\mu\nu}(Z^\alpha_i, \bar{Z}^\alpha_i)\partial_\mu \wedge \partial_\nu \qquad \mu, \nu \sim (\alpha, i). \tag{38}$$

After some lengthy algebra it is found that the only non-vanishing brackets of the vertex variables (35) are the following

$$\begin{aligned}
&\{Z^{i+1}_1, Z^i_1\} = \frac{i}{2} Z^{i+1}_2 \bar{Z}^i_2 - \frac{i}{8}\Lambda^i\left(Z^{i+1}_1 \bar{Z}^i_2 - Z^{i+1}_2 Z^i_1\right), \\
&\{Z^{i+1}_1, Z^i_2\} = -\frac{i}{2} Z^{i+1}_2 \bar{Z}^i_1 + \frac{i}{8}\Lambda^i\left(Z^{i+1}_1 \bar{Z}^i_1 + Z^{i+1}_2 Z^i_2\right), \\
&\{Z^{i+1}_1, \bar{Z}^i_1\} = -\frac{i}{8}\Lambda^i\left(Z^{i+1}_1 Z^i_2 + Z^{i+1}_2 \bar{Z}^i_1\right), \\
&\{Z^{i+1}_1, \bar{Z}^i_2\} = -\{Z^{i+1}_2, Z^i_1\}, \\
\\
&\{Z^{i+1}_2, Z^i_1\} = -\frac{i}{8}\Lambda^i\left(Z^{i+1}_1 Z^i_1 - Z^{i+1}_2 \bar{Z}^i_2\right), \\
&\{Z^{i+1}_2, Z^i_2\} = \{Z^{i+1}_1, \bar{Z}^i_1\}, \\
&\{Z^{i+1}_2, \bar{Z}^i_1\} = \frac{i}{2} Z^{i+1}_1 Z^i_2 + \frac{i}{8}\Lambda^i\left(Z^{i+1}_1 \bar{Z}^i_1 + Z^{i+1}_2 Z^i_2\right), \\
&\{Z^{i+1}_2, \bar{Z}^i_2\} = -\frac{i}{2} Z^{i+1}_1 Z^i_1 + \frac{i}{8}\Lambda^i\left(Z^{i+1}_1 \bar{Z}^i_2 - Z^{i+1}_2 Z^i_1\right), \\
\\
&\{Z_1, \bar{Z}_1\}^{i+1} = \frac{i}{8}\,\Lambda^i\,\left(Z_1\bar{Z}_2 + \bar{Z}_1 Z_2\right)^{i+1} + \frac{i}{8}\,\Lambda^{i+1}\,\left(Z_1 Z_2 + \bar{Z}_1\bar{Z}_2\right)^{i+1}, \\
&\{Z_1, Z_2\}^{i+1} = \frac{i}{2}\left(Z_1 Z_2\right)^{i+1} - \frac{i}{8}\Lambda^{i+1} - \frac{i}{8}\Lambda^i\left(Z^2_1 - Z^2_2\right)^{i+1}, \\
&\{Z_1, \bar{Z}_2\}^{i+1} = -\frac{i}{2}\left(Z_1\bar{Z}_2\right)^{i+1} - \frac{i}{8}\Lambda^i - \frac{i}{8}\Lambda^{i+1}\left(Z^2_1 - \bar{Z}^2_2\right)^{i+1}, \\
&\{Z_2, \bar{Z}_2\}^{i+1} = i\left|Z^{i+1}_1\right|^2 + \frac{i}{8}\,\Lambda^i\,\left(Z_1\bar{Z}_2 + \bar{Z}_1 Z_2\right)^{i+1} - \frac{i}{8}\,\Lambda^{i+1}\,\left(Z_1 Z_2 + \bar{Z}_1\bar{Z}_2\right)^{i+1}.
\end{aligned} \tag{39}$$

where the parameter Λ^i is real (ie., $\Lambda^i = \bar{\Lambda}^i$) and is defined by the dual form in terms of the vertex variables either at the i^{th} or at the $i+1^{th}$ vertex[1]. That is,

$$\Lambda^i = \left(\frac{\bar{Z}_1{}^2 - Z_1^2 + \bar{Z}_2{}^2 - Z_2^2}{\bar{Z}_1 Z_2 - Z_1 \bar{Z}_2}\right)^i \tag{40}$$

$$= \left(\frac{\bar{Z}_1{}^2 - Z_1^2 - \bar{Z}_2{}^2 + Z_2^2}{Z_1 Z_2 - \bar{Z}_1 \bar{Z}_2}\right)^{i+1} . \tag{41}$$

Furthermore, one can check that the following identities are satisfied

$$\begin{aligned}
\{|Z_1|^2 + |Z_2|^2, Z_1\}^i &= \{|Z_1|^2 + |Z_2|^2, \bar{Z}_1\}^i = 0,\\
\{|Z_1|^2 + |Z_2|^2, Z_2\}^i &= \{|Z_1|^2 + |Z_2|^2, \bar{Z}_2\}^i = 0,\\
\{\left(|Z_1|^2 + |Z_2|^2\right)^{i+1}, Z_1^i\} &= \{\left(|Z_1|^2 + |Z_2|^2\right)^{i+1}, \bar{Z}_1^i\} = 0,\\
\{\left(|Z_1|^2 + |Z_2|^2\right)^{i+1}, Z_2^i\} &= \{\left(|Z_1|^2 + |Z_2|^2\right)^{i+1}, \bar{Z}_2^i\} = 0,\\
\{\left(|Z_1|^2 + |Z_2|^2\right)^{i}, Z_1^{i+1}\} &= \{\left(|Z_1|^2 + |Z_2|^2\right)^{i}, \bar{Z}_1^{i+1}\} = 0,\\
\{\left(|Z_1|^2 + |Z_2|^2\right)^{i}, Z_2^{i+1}\} &= \{\left(|Z_1|^2 + |Z_2|^2\right)^{i}, \bar{Z}_2^{i+1}\} = 0.
\end{aligned}$$

Thus $|Z_1^i|^2 + |Z_2^i|^2$ are Casimir elements of the derived algebra (39). [Note that, this result is expected, due to the form of the vertex variables defined in (35)].

To sum up, the relations (39) determine a closed, albeit nonlinear, Poisson bracket algebra that obeys the Jacobi identity and the Leibnitz rule, as can be concluded either by general arguments or by explicit evaluation of the Schouten bracket of the pertinent Poisson bi-vector (38). In particular, the Poisson brackets (39) determine a Poisson structure that is a proper descendant of the initial SU(2) Lie-Poisson structure. The construction can be extended to all levels of the hierarchy in a self-similar way as explained above. Therefore, an infinite hierarchy of Poisson structures as descendants of the SU(2) Lie algebra can be constructed.

V. CONCLUDING REMARKS

In conclusion, it has been shown here that in the case of a piecewise linear polygonal string the SU(2) Lie-Poisson structure gives rise to an infinite hierarchy of Poisson structures, as its descendants. Each level of Poisson structures engages an increasingly number of vertices along the string, thus they are different. It has been shown by an explicit construction of the first level descendant, that the spinor representation of the Lie-Poisson bracket is a computationally tractable realisation. The novel Poisson structure that has been constructed explicitly, engages a chain of four vertices along the string (three links), and the higher level descendants engage an increasing number of vertices.

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Appendix A: Link Vs Vertex Variables

Directly from (35) the following systems are also satisfied

$$\begin{aligned} z_1^{i+1} &= z_1^i\, Z_1^{i+1} - \bar{z}_2^i\, Z_2^{i+1} \\ z_2^{i+1} &= z_2^i\, Z_1^{i+1} + \bar{z}_1^i\, Z_2^{i+1} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} z_1^i &= \left(z_1\, \bar{Z}_1 + \bar{z}_2\, Z_2\right)^{i+1} \\ z_2^i &= \left(z_2\, \bar{Z}_1 - \bar{z}_1\, Z_2\right)^{i+1}, \end{aligned} \tag{A1}$$

[1] This is proven in the Appendix A; due to (A2)

where $|z_1^i|^2 + |z_2^i|^2 = 1$. Note that, by definition due to (29) the link variables satisfy the identity $(z_1 z_2)^i \equiv (\bar{z}_1 \bar{z}_2)^i$ which is not true for the vertex variables.

Due to (A1) it is easy to prove that

$$\left(\frac{|z_1|^2 - |z_2|^2}{z_1 z_2}\right)^{i+1} = \left(\frac{\bar{Z}_1{}^2 - Z_1^2 + \bar{Z}_2{}^2 - Z_2^2}{\bar{Z}_1 Z_2 - Z_1 \bar{Z}_2}\right)^{i+1}, \qquad \left(\frac{|z_1|^2 - |z_2|^2}{z_1 z_2}\right)^{i} = \left(\frac{\bar{Z}_1{}^2 - Z_1^2 - \bar{Z}_2{}^2 + Z_2^2}{Z_1 Z_2 - \bar{Z}_1 \bar{Z}_2}\right)^{i+1}. \quad \text{(A2)}$$

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