Self-duality, integrability, topological entanglement and a field theory model of two linked rings

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Self-duality, integrability, topological entanglement and a field theory model of two linked rings

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Abstract. This paper examines the link between the statistical mechanics of polymer systems and field theory, focusing on topologically linked polymer rings in a 4-plat configuration. We derive the partition function and interpret it through the lens of physics of anyons. Using techniques from theory of anyons, we establish the self-duality equations and find solutions that minimize the system's energy. Under specific conditions, these self-duality equations simplify to the Euclidean sinh-Gordon, cosh-Gordon, or Liouville equations, for which we compute translationally invariant solutions and corresponding polymer densities. Our findings provide new insights into the field-theoretic approach to topologically constrained polymers.

1 Introduction

In the 1970s, R.G. de Gennes discovered a profound connection between the statistical mechanics of long polymer molecules and magnetic systems described by multicomponent complex field theories with $O(N)$ symmetry [1, 2]. This groundbreaking discovery paved the way for a new field of research and ultimately earned him the Nobel Prize in Physics in 1991. The idea of applying field theory techniques to study the properties of polymers was further developed by many authors, including [3, 4, 5]. One of the primary goals was to establish a theoretical framework for investigating topologically linked polymers [6]. This research program was realized in the specific case of polymer rings linked together to form a $2s$-plat, as described in Refs. [7, 8, 9]. A $2s$-plat is a link whose path in space is characterized by a fixed number of $2s$ maxima and minima along a specified direction, referred to here as the height (e.g., the z-axis). The aim of this paper is to present this line of research and our latest findings to a broader audience. We will focus on describing the simplest case, where $s = 2$, which has been thoroughly studied in [9].

This paper is organized as follows. In Section 2, we formulate the problem and, following the approach in [9], derive the partition function for a system of two linked loops in a topological state characterized by a fixed Gauss linking number. In the field-theoretic framework, the polymer partition function is interpreted as a correlation function for a mixture of four types of anyons. In Section 3, utilising methods from the physics of anyons, we derive the self-duality conditions and identify their solutions that minimize the

system's energy. In Section 4, we demonstrate that the self-dual equations reduce to the two-dimensional Euclidean sinh-Gordon or cosh-Gordon equations, depending on the sign of the integration constant. We also show that when the integration constant is zero, the equations simplify to the Liouville equation. Furthermore, we compute the translationally invariant solutions of the Euclidean sinh-Gordon and cosh-Gordon equations and express the polymer densities in terms of these solutions. Finally, in Section 5, we present our conclusions.

2 A field-theoretic model of two concatenated polymer rings

In this section, we aim to formulate a field-theoretic model involving two concatenated rings forming a 4-plat. A 4-plat knot refers to a type of braid where two strands cross over each other multiple times before returning to their original positions, and when closed, the structure resembles a knot. In our case, the two strands form two rings, making the system a complex topological structure. Our goal is to

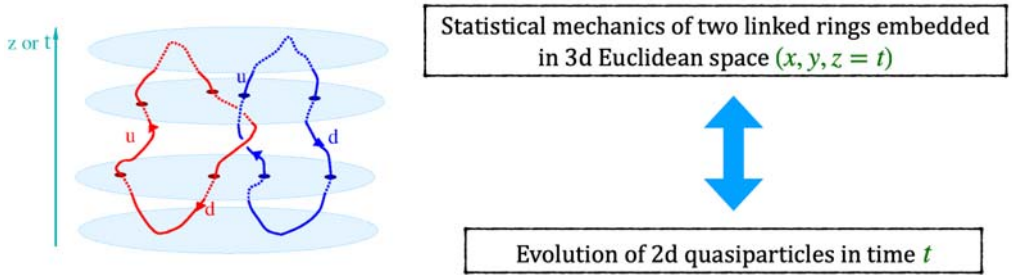

Figure 1: Polymer rings vs. quasiparticles.

construct statistical mechanics (statistical field theory) of such objects, i.e., linked (polymer) rings with a given topological state. A natural question arises: is this even possible? Our approach to constructing such a system draws on the physics of anyons, a type of quasiparticle that emerges in two-dimensional systems. Anyons are closely tied to topological quantum field theories (TQFTs) and topological phases of matter. Their defining characteristic is that their quantum state depends on the braiding of particle trajectories, meaning how these particles wind around one another in two-dimensional space. Given this, it is natural to interpret the three-dimensional trajectories of two-dimensional quasiparticles as paths of entangled polymers (*cf.* Fig.1). This opens up the possibility of applying TQFT methods to describe such systems. However, one key limitation must be considered: the conformations of the rings must each exhibit one minimum and one maximum. This constraint is crucial for the validity of the approach.

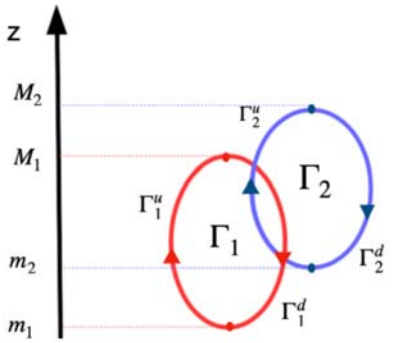

Figure 2: The 4-plat within the framework of our parameterization.

More precisely, we consider two loops having two points of minima and maxima along the z-direction (*cf.* Fig.2). The topological relations are fixed using the Gauss linking number (GLN),

$$\chi(\Gamma_1,\Gamma_2) \;=\; W_{\Gamma_1^u\Gamma_2^d}(z_0,z_1)+W_{\Gamma_1^u\Gamma_2^u}(z_0,z_1)+W_{\Gamma_1^d\Gamma_2^u}(z_0,z_1)+W_{\Gamma_1^d\Gamma_2^d}(z_0,z_1) \tag{1}$$

which is expressed by the winding number $W_{\Gamma\,\Gamma'}\,(z_0,z_1)$ of two monotonic curves Γ, Γ' between the two heights z_0 and z_1,

$$W_{\Gamma\Gamma'}\,(z_0,z_1) \;=\; \epsilon_{ij}\int_{z_0}^{z_1}\mathrm{d}\big(x^i(z)-x'^i(z)\big)\,\frac{\big(x^j(z)-x'^j(z)\big)}{\left|\boldsymbol{x}(z)-\boldsymbol{x}'(z)\right|^2},\qquad i,j=1,2. \tag{2}$$

The partition function $Z(\mu)$ of the system composed by the two linked loops Γ_1, Γ_2 may be written as follows,

$$Z(\mu) \;=\; \left[\prod_{a=1}^{2}\int_{\boldsymbol{x}_a(m_a)}^{\boldsymbol{x}_a(M_a)}\mathcal{D}\boldsymbol{x}_a^u(z)\int_{\boldsymbol{x}_a(M_a)}^{\boldsymbol{x}_a(m_a)}\mathcal{D}\boldsymbol{x}_a^d(z)\right]\delta\left(\chi(\Gamma_1,\Gamma_2)-\mu\right)\mathrm{e}^{-A_{\mathsf{pol}}}\,. \tag{3}$$

The coordinates $\boldsymbol{x}_a(m_a)$ and $\boldsymbol{x}_a(M_a)$ denoting respectively the locations of the points of maximal and minimal height of Γ_a are fixed. Moreover, the topological constraint

$$\chi(\Gamma_1,\Gamma_2) \;=\; \mu \tag{4}$$

with μ being a constant is imposed in Eq. (3) using a Dirac delta function. Finally, A_{pol} is the term associated to chain connectivity:

$$A_{\mathsf{pol}} \;=\; \prod_{a=1}^{2}\int_{m_a}^{M_a}\mathrm{d}z\left[g_{a,u}\left|\frac{\mathrm{d}\boldsymbol{x}_a^u(z)}{\mathrm{d}z}\right|^2+g_{a,d}\left|\frac{\mathrm{d}\boldsymbol{x}_a^d(z)}{\mathrm{d}z}\right|^2\right]. \tag{5}$$

For simplicity, interactions have been omitted, though this approach can be easily extended to include the excluded volume potential. In the expressions above, $\boldsymbol{x}_a^{u,d}(z)$ represent the curves describing the paths of $\Gamma_a^{u,d}$. The constants $g_{a,u}$ and $g_{a,d}$ are related to the Kuhn length and characterize the flexibility of the chains Γ_a^u and Γ_a^d.

At this stage, the Fourier transform can be applied to express the Dirac delta function $\delta\left(\chi(\Gamma_1,\Gamma_2)-\mu\right)$ in the following form:

$$\delta\left(\chi(\Gamma_1,\Gamma_2)-\mu\right) \;=\; \int_{-\infty}^{+\infty}\mathrm{d}\lambda\;\mathrm{e}^{\mathrm{i}\lambda\mu}\;\mathrm{e}^{-\mathrm{i}\lambda\chi(\Gamma_1,\Gamma_2)}. \tag{6}$$

This allows to rewrite the partition function $Z(\mu)$ in the simpler form:

$$Z(\mu) \;=\; \int_{-\infty}^{+\infty}\mathrm{d}\lambda\;\mathrm{e}^{\mathrm{i}\lambda\mu}\;Z(\lambda), \tag{7}$$

where

$$Z(\lambda) \;=\; \left[\prod_{a=1}^{2}\int_{\boldsymbol{x}_a(m_a)}^{\boldsymbol{x}_a(M_a)}\mathcal{D}\boldsymbol{x}_a^u(z)\int_{\boldsymbol{x}_a(M_a)}^{\boldsymbol{x}_a(m_a)}\mathcal{D}\boldsymbol{x}_a^d(z)\right]\mathrm{e}^{-A_{\mathsf{pol}}}\,\mathrm{e}^{-\mathrm{i}\lambda\chi(\Gamma_1,\Gamma_2)}\,. \tag{8}$$

The Fourier transformation from $Z(\mu)$ to $Z(\lambda)$ is analogous to the transition from the microcanonical ensemble to the canonical ensemble, but here the role of the Hamiltonian H is replaced by the Gauss linking number $\chi(\Gamma_1,\Gamma_2)$, and the Boltzmann factor $\beta=(kT)^{-1}$ is substituted by $\mathrm{i}\lambda$.

As shown in Ref. [8], the exponential $\mathrm{e}^{-\mathrm{i}\lambda\chi(\Gamma_1,\Gamma_2)}$, which involves a highly complex dependence on the conformations $\boldsymbol{x}_a^{u,d}(z)$, can be simplified by rewriting it as the partition function of an abelian BF model:

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}\lambda\chi(\Gamma_1,\Gamma_2)} &= \int \mathcal{D}\boldsymbol{B}_1(\boldsymbol{x},t)\mathcal{D}\boldsymbol{B}_2(\boldsymbol{x},t)\mathcal{D}B_1^0(\boldsymbol{x},t)\mathcal{D}B_2^0(\boldsymbol{x},t)\\ &\times\exp\left\{-\mathrm{i}S_{\mathsf{BF}}-\mathrm{i}\lambda\int \mathrm{d}^2x\mathrm{d}t\left[\boldsymbol{B}_2\cdot\boldsymbol{J}_1+B_2^0J_1^0\right]-\frac{\mathrm{i}\kappa}{8\pi^2}\int \mathrm{d}^2x\mathrm{d}t\left[\boldsymbol{B}_1\cdot\boldsymbol{J}_2+B_1^0J_2^0\right]\right\},\end{aligned} \tag{9}$$

where the BF action S_{BF} is given by:

$$S_{\mathsf{BF}} = \frac{\kappa}{4\pi}\epsilon_{ij}\int \mathrm{d}^2x\mathrm{d}t\left[B_1^0\partial^i B_2^j+B_2^0\partial^i B_1^j\right], \qquad i,j=1,2. \tag{10}$$

Here and throughout, we assume the summation convention for repeated upper and lower indices representing spatial coordinates. Let us note that the components $(\boldsymbol{B}_a, B_a^0)$, $a=1,2$ of the BF fields are coupled in Eq. (9) to certain imaginary currents flowing through the loops Γ_1 and Γ_2. The vector components $\boldsymbol{B}_a(\boldsymbol{x},t)$ satisfy the Coulomb gauge condition $\boldsymbol{\nabla}\cdot\boldsymbol{B}_a(\boldsymbol{x},t)=0$.

Applying the identity (9) it is possible to convert the partition function $Z(\lambda)$ to the following form:

$$Z(\lambda) = \int\prod_{a=1}^{2}\mathcal{D}\boldsymbol{B}_a\mathcal{D}B_a^0\, Z_a^u(\lambda)\, Z_a^d(\lambda)\,\mathrm{e}^{-\mathrm{i}S_{\mathsf{BF}}} \tag{11}$$

where

$$Z_a^u(\lambda) = \int_{\boldsymbol{x}_a(m_a)}^{\boldsymbol{x}_a(M_a)}\mathcal{D}\boldsymbol{x}_a^u(z)\,\mathrm{e}^{-S_a^u}, \tag{12}$$

$$Z_a^d(\lambda) = \int_{\boldsymbol{x}_a(M_a)}^{\boldsymbol{x}_a(m_a)}\mathcal{D}\boldsymbol{x}_a^d(z)\,\mathrm{e}^{-S_a^d} \tag{13}$$

and

$$S_a^u = \int_{m_a}^{M_a}\mathrm{d}z\left[g_{a,u}\left|\frac{\mathrm{d}\boldsymbol{x}_a^u(z)}{\mathrm{d}z}\right|^2+\mathrm{i}\sum_{b=1}^{2}C_{ab}\left(\frac{\mathrm{d}\boldsymbol{x}_a^u(z)}{\mathrm{d}z}\cdot\boldsymbol{B}_b(\boldsymbol{x}_a^u(z),z)+B_b^0(\boldsymbol{x}_a^u(z),z)\right)\right], \tag{14}$$

$$S_a^d = \int_{m_a}^{M_a}\mathrm{d}z\left[g_{a,d}\left|\frac{\mathrm{d}\boldsymbol{x}_a^d(z)}{\mathrm{d}z}\right|^2-\mathrm{i}\sum_{b=1}^{2}C_{ab}\left(\frac{\mathrm{d}\boldsymbol{x}_a^d(z)}{\mathrm{d}z}\cdot\boldsymbol{B}_b(\boldsymbol{x}_a^d(z),z)+B_b^0(\boldsymbol{x}_a^d(z),z)\right)\right]. \tag{15}$$

The 2×2 matrix C_{ab} is given by

$$C_{ab} = \begin{bmatrix} 0 & \lambda \\ \frac{\kappa}{8\pi^2} & 0 \end{bmatrix}.$$

Let us note S_a^u and S_a^d are formally equal to the actions of two particles immersed in the magnetic fields generated by the vector potentials $\boldsymbol{B}_1$, $\boldsymbol{B}_2$ and interacting with the external potentials B_1^0, B_2^0. Accordingly, $Z_a^u(\lambda)$ may be interpreted as the transition amplitudes of particles $\boldsymbol{x}_a^u(z)$ to pass from an initial state $|\,\boldsymbol{x}_a^u(m_a)\,\rangle$ to a final state $\langle\,\boldsymbol{x}_a^u(M_a)\,|$. An analogous interpretation can be given to $Z_a^d(\lambda)$. This analogy with quantum mechanics allows to pass from paths to fields using the procedure of second quantisation.

Indeed, the key tool here is the correspondence referred to as the quantum-mechanical analogy. Specifically, one can introduce the probability distribution $G(\mathbf{r}_1,\mathbf{r}_0,L;\eta)$, which describes the likelihood of a fluctuating polymer chain having one end at $\mathbf{r}_0$ and the other at $\mathbf{r}_1$. This distribution depends on the

polymer length L and an auxiliary field η, often referred to as white noise, which allows the implementation of self-avoiding interactions in the path integral.[1] The probability distribution $G(\mathbf{r}_1, \mathbf{r}_0, L; \eta)$ can be interpreted as a quantum-mechanical transition amplitude, describing the evolution from the state $|\mathbf{r}_0\rangle$ at time $s = 0$ to the state $\langle \mathbf{r}_1|$ at time $s = L$, in purely imaginary time $s = it$. In the special case of a circular polymer, where $\mathbf{r}_1 = \mathbf{r}_0$, this simplifies. The distribution $G(\mathbf{r}_1, \mathbf{r}_0, L; \eta)$

i. is given by

$$G(\mathbf{r}_1, \mathbf{r}_0, L; \eta) = \langle \mathbf{r}_1 | \mathrm{e}^{-\int_0^L \mathrm{d}t\, \mathcal{H}(\mathbf{p}(s), \mathbf{r}(s); \eta)} | \mathbf{r}_0 \rangle,$$

where $\mathcal{H}$ is the Hamiltonian, and $\mathbf{p}(s) = \dot{\mathbf{r}}(s) = \mathrm{d}\mathbf{r}(s)/\mathrm{d}s$;

ii. satisfies the pseudo-Schrödinger equation:

$$\left[\frac{\partial}{\partial L} + \mathcal{H}(-\boldsymbol{\nabla}_1, \mathbf{r}_0; \eta)\right] G(\mathbf{r}_1, \mathbf{r}_0, L; \eta) = \delta(\mathbf{r}_1 - \mathbf{r}_0)\, \delta(L). \tag{16}$$

The solution to the Eq. (16) can be written as a path integral over replica complex scalar fields $\boldsymbol{\Psi}^*$ and $\boldsymbol{\Psi}$, where $\boldsymbol{\Psi} = (\psi^{(1)}, \ldots, \psi^{(n)})$, and in the limit $n \to 0$, one gets

$$\begin{aligned} G(\mathbf{r}_1, \mathbf{r}_0, L; \eta) &= \lim_{n\to 0} \int \mathcal{D}\boldsymbol{\Psi}^*(\boldsymbol{x}, s) \mathcal{D}\boldsymbol{\Psi}(\boldsymbol{x}, s)\, \psi^{*(1)}(\mathbf{r}_1, L) \psi^{(1)}(\mathbf{r}_0, 0) \\ &\times \mathrm{e}^{-\int \mathrm{d}^3x \int_0^L \mathrm{d}s \left[\boldsymbol{\Psi}^*(\boldsymbol{x},s)\left(\partial_s - a/6\boldsymbol{\nabla}^2 - i\sqrt{v_0}\eta(\boldsymbol{x})\right)\boldsymbol{\Psi}(\boldsymbol{x},s)\right]}. \end{aligned}$$

Using the quantum-mechanical analogy, one can demonstrate that the "one-particle transition amplitudes",

$$Z_a^{u,d}(\lambda) = G_a^{u,d}(\boldsymbol{x}_a(M_a) - \boldsymbol{x}_a(m_a), M_a - m_a)\,, \tag{17}$$

satisfy pseudo-Schrödinger equations:

$$\begin{aligned} &\left[\frac{\partial}{\partial t} - \mathrm{i}\sum_{b=1}^{2} C_{ab} B_b^0(\boldsymbol{x}, t) - \frac{1}{4g_{a,u}}\left(\boldsymbol{\nabla}_{\boldsymbol{x}} - \mathrm{i}\sum_{b=1}^{2} C_{ab}\boldsymbol{B}_b(\boldsymbol{x}, t)\right)^2\right] G_a^u(\boldsymbol{x} - \boldsymbol{y}, t - t') \\ &\qquad = \delta^{(2)}(\boldsymbol{x} - \boldsymbol{y})\, \delta(t - t')\,, \end{aligned} \tag{18}$$

$$\begin{aligned} &\left[\frac{\partial}{\partial t} + \mathrm{i}\sum_{b=1}^{2} C_{ab} B_b^0(\boldsymbol{x}, t) - \frac{1}{4g_{a,d}}\left(\boldsymbol{\nabla}_{\boldsymbol{x}} + \mathrm{i}\sum_{b=1}^{2} C_{ab}\boldsymbol{B}_b(\boldsymbol{x}, t)\right)^2\right] G_a^d(\boldsymbol{x} - \boldsymbol{y}, t - t') \\ &\qquad = \delta^{(2)}(\boldsymbol{x} - \boldsymbol{y})\, \delta(t - t')\,. \end{aligned} \tag{19}$$

Then, by employing well-established field-theoretic techniques, one can derive

$$\begin{aligned} Z(\lambda) &= \lim_{n\to 0} \int \prod_{a=1}^{2} \mathcal{D}\boldsymbol{B}_a \mathcal{D}B_a^0 \mathcal{D}\boldsymbol{\Psi}_a^{*\,u} \mathcal{D}\boldsymbol{\Psi}_a^u \mathcal{D}\boldsymbol{\Psi}_a^{*\,d} \mathcal{D}\boldsymbol{\Psi}_a^d \\ &\times \psi_a^{*\,u(1)}(\boldsymbol{x}(M_a), M_a)\, \psi_a^{u(1)}(\boldsymbol{x}(m_a), m_a)\, \psi_a^{*\,d(1)}(\boldsymbol{x}(m_a), m_a)\, \psi_a^{d(1)}(\boldsymbol{x}(M_a), M_a) \\ &\times \mathrm{e}^{-S_{\mathsf{matter}}}\, \mathrm{e}^{-\mathrm{i}S_{\mathsf{BF}}}, \end{aligned} \tag{20}$$

[1] These interactions are expressed as

$$\mathrm{e}^{-v_0 \int_0^L \mathrm{d}s \int_0^L \mathrm{d}s' \delta(\mathbf{r}(s) - \mathbf{r}(s'))} = \int \mathcal{D}\eta(\boldsymbol{x})\, \exp\left\{-\int \mathrm{d}^3x \frac{\eta^2}{4} - \mathrm{i}v_0 \int_0^L \mathrm{d}s\, \eta(\mathbf{r}(s))\right\},$$

where $\mathbf{r}(s)$ is the polymer path, and s is the arc-length.

where the action S_{matter}, after integrating out the z-components B_a^0, takes the form[2]

$$S_{\mathsf{matter}} = \sum_{a=1}^{2}\int \mathrm{d}^2x \int_0^T \mathrm{d}t \left[\boldsymbol{\Psi}_a^{*\,u}\partial_0\boldsymbol{\Psi}_a^u + \frac{1}{4g_{a,u}}\left|\left(\boldsymbol{\nabla}_{\boldsymbol{x}} - \mathrm{i}\sum_{b=1}^{2} C_{ab}\boldsymbol{B}_b\right)\boldsymbol{\Psi}_a^u\right|^2 + \boldsymbol{\Psi}_a^{*\,d}\partial_0\boldsymbol{\Psi}_a^d + \frac{1}{4g_{a,d}}\left|\left(\boldsymbol{\nabla}_{\boldsymbol{x}} + \mathrm{i}\sum_{b=1}^{2} C_{ab}\boldsymbol{B}_b\right)\boldsymbol{\Psi}_a^d\right|^2\right]. \tag{21}$$

As a final remark in this section, we note that although the partition function $Z(\lambda)$ in Eq. (20) originates from a polymer problem, it can also be interpreted as the correlation function of a mixture of four types of anyon particles, with densities

$$\left|\boldsymbol{\Psi}_a^{u,d}\right|^2 := \sum_{r=1}^{n} \psi_a^{u,d(r)}\psi_a^{*\,u,d(r)},$$

and the action (21).

3 Self-duality in the system of two concatenated polymer rings

The analogy with anyons suggests that the action (21) should be examined using the methods of self-dual systems. To this end, following [10, 11] and [7], we introduce the covariant derivatives:

$$\mathsf{D}_{a,j}^u = \partial_j - \mathrm{i}\sum_{b=1}^{2} C_{ab}\boldsymbol{B}_{b,j}, \tag{22}$$

$$\mathsf{D}_{a,j}^d = \partial_j + \mathrm{i}\sum_{b=1}^{2} C_{ab}\boldsymbol{B}_{b,j}, \tag{23}$$

where $j = 1, 2$ labels the spatial coordinates and $a = 1, 2$ labels the contributions coming from loops Γ_1 and Γ_2. As a result, the action in Eq. (21) can be rewritten as follows:

$$S_{\mathsf{matter}} = I_{\mathsf{T}} + I_{\mathsf{sd}} + I_{\mathsf{C}}, \tag{24}$$

where

— I_{T} is the "time" dependent part,

$$I_{\mathsf{T}} = \sum_{a=1}^{2}\int \mathrm{d}^2x \int_0^T \mathrm{d}t \left[\boldsymbol{\Psi}_a^{*\,u}\partial_0\boldsymbol{\Psi}_a^u + \boldsymbol{\Psi}_a^{*\,d}\partial_0\boldsymbol{\Psi}_a^d\right]; \tag{25}$$

— I_{sd} is the self-dual part of the action,

$$I_{\mathsf{sd}} = \sum_{a=1}^{2}\int \mathrm{d}^2x \int_0^T \mathrm{d}t \left[\frac{1}{4g_{a,u}}\left|\left(\mathsf{D}_{a,1}^u + \mathrm{i}\mathsf{D}_{a,2}^u\right)\boldsymbol{\Psi}_a^u\right|^2 + \frac{1}{4g_{a,d}}\left|\left(\mathsf{D}_{a,1}^d + \mathrm{i}\mathsf{D}_{a,2}^d\right)\boldsymbol{\Psi}_a^d\right|^2\right]; \tag{26}$$

[2] For simplicity, from now on we assume that $m_1 = m_2 = 0$ and $M_1 = M_2 = T$. Integrating out B_a^0 one gets the constraint

$$\sum_{c=1}^{2} d_{ac}\epsilon^{ij}\partial_i B_{c,j} = \sum_{b=1}^{2} C_{ba}\left(-|\boldsymbol{\Psi}_b^u|^2 + \left|\boldsymbol{\Psi}_b^d\right|^2\right),$$

where $d_{ab} = \begin{bmatrix} 0 & \frac{\kappa}{4\pi} \\ \frac{\kappa}{4\pi} & 0 \end{bmatrix}$.

— I_{C} accounts for the Coulomb-like interactions,

$$I_{\mathsf{C}} = \frac{\lambda}{8\pi}\int \mathrm{d}^2x \int_0^T \mathrm{d}t \left[\left(-\frac{1}{g_{1,u}}\left|\mathbf{\Psi}_1^u\right|^2 + \frac{1}{g_{1,d}}\left|\mathbf{\Psi}_1^d\right|^2\right)\left(-\left|\mathbf{\Psi}_2^u\right|^2 + \left|\mathbf{\Psi}_2^d\right|^2\right) + \left(\frac{1}{g_{2,u}}\left|\mathbf{\Psi}_2^u\right|^2 - \frac{1}{g_{2,d}}\left|\mathbf{\Psi}_2^d\right|^2\right)\left(-\left|\mathbf{\Psi}_1^u\right|^2 + \left|\mathbf{\Psi}_1^d\right|^2\right)\right]. \tag{27}$$

In writing the action in Eqs. (24)-(27) terms that are total derivatives have been neglected.

It turns out that the term I_{T} becomes negligibly small when the height τ, in which the paths of both loops are defined, become large. To demonstrate this, we can perform the variable change $\tau\sigma = t$ in Eqs. (25), (26), and (27). This transformation causes the self-dual contribution I_{sd} and the Coulomb interaction terms I_{C} to acquire a factor of τ, while I_{T} remains unaffected. Henceforth, we will work in the limit of large τ, where

$$S_{\mathsf{matter}} \sim I_{\mathsf{sd}} + I_{\mathsf{C}}\,. \tag{28}$$

Notably, while the variable t represents the height of the polymer loops rather than time, the action remains formally equivalent to that of a system of anyons. This leads to the expectation that, in the limit $\tau \to \infty$, the monomer distribution will not vary significantly at different heights, allowing us to discuss static solutions akin to those for anyons — albeit with the distinction that "static" here refers to an absence of changes along the z-axis.

Proceeding analogously as in the case of anyons, on the basis of Eq. (28) one can define the density of energy per unit of height z,

$$\mathcal{E}(z) = \sum_{a=1}^{2}\int \mathrm{d}^2x \left[\frac{1}{4g_{a,u}}\left|\left(\mathsf{D}_{a,1}^u + \mathrm{i}\mathsf{D}_{a,2}^u\right)\mathbf{\Psi}_a^u\right|^2 + \frac{1}{4g_{a,d}}\left|\left(\mathsf{D}_{a,1}^d + \mathrm{i}\mathsf{D}_{a,2}^d\right)\mathbf{\Psi}_a^d\right|^2\right] + \frac{\lambda}{8\pi}\int \mathrm{d}^2x \left[\left(-\frac{1}{g_{1,u}}\left|\mathbf{\Psi}_1^u\right|^2 + \frac{1}{g_{1,d}}\left|\mathbf{\Psi}_1^d\right|^2\right)\left(-\left|\mathbf{\Psi}_2^u\right|^2 + \left|\mathbf{\Psi}_2^d\right|^2\right) + \left(\frac{1}{g_{2,u}}\left|\mathbf{\Psi}_2^u\right|^2 - \frac{1}{g_{2,d}}\left|\mathbf{\Psi}_2^d\right|^2\right)\left(-\left|\mathbf{\Psi}_1^u\right|^2 + \left|\mathbf{\Psi}_1^d\right|^2\right)\right]. \tag{29}$$

An interesting case is when the monomer densities of Γ_a^u or Γ_a^u can be considered as constant. For instance, assuming that $|\mathbf{\Psi}_1^d|^2 = V_1^2$ and $|\mathbf{\Psi}_2^d|^2 = V_2^2$ with $V_1, V_2 = \text{const.}$, we obtain

$$\mathcal{E}(z) = \mathcal{E}_{\mathsf{sd}}(z) + \frac{\lambda}{8\pi}\int \mathrm{d}^2x \left[\left(\frac{1}{g_{1,u}}\left|\mathbf{\Psi}_1^u\right|^2 - \frac{1}{g_{1,d}}V_1^2\right)\left(\left|\mathbf{\Psi}_2^u\right|^2 - V_2^2\right) + \left(\frac{1}{g_{2,u}}\left|\mathbf{\Psi}_2^u\right|^2 - \frac{1}{g_{2,d}}V_2^2\right)\left(-\left|\mathbf{\Psi}_1^u\right|^2 + V_1^2\right)\right], \tag{30}$$

where $\mathcal{E}_{\mathsf{sd}}(z)$ is the energy density of the self-dual part,

$$\mathcal{E}_{\mathsf{sd}}(z) = \sum_{a=1}^{2}\int \mathrm{d}^2x \left[\frac{1}{4g_{a,u}}\left|\left(\mathsf{D}_{a,1}^u + \mathrm{i}\mathsf{D}_{a,2}^u\right)\mathbf{\Psi}_a^u\right|^2 + \frac{1}{4g_{a,d}}\left|\left(\mathsf{D}_{a,1}^d + \mathrm{i}\mathsf{D}_{a,2}^d\right)\mathbf{\Psi}_a^d\right|^2\right]. \tag{31}$$

The energy in Eq. (30) is minimized by the self-duality conditions:

$$\left[\mathsf{D}_{a,1}^{u,d} + \mathrm{i}\mathsf{D}_{a,2}^{u,d}\right]\mathbf{\Psi}_a^{u,d} = 0 \tag{32}$$

which are satisfied. There are two distinct minima corresponding to the following cases:

$$|\mathbf{\Psi}_1^u|^2 = \frac{g_{1,u}}{g_{1,d}}V_1^2 \quad \text{and} \quad |\mathbf{\Psi}_2^u|^2 = \frac{g_{2,u}}{g_{2,d}}V_2^2 \tag{33}$$

or

$$|\mathbf{\Psi}_1^u|^2 \;=\; V_1^2 \quad \text{and} \quad |\mathbf{\Psi}_2^u|^2 \;=\; V_2^2. \tag{34}$$

Most interesting is probably the homopolymer case in which all legs $\Gamma_1^{u,d}$ and $\Gamma_2^{u,d}$ are homogeneous, so that

$$\frac{1}{g_{1,u}} \;=\; \frac{1}{g_{1,d}} \;=\; \frac{1}{g_{2,u}} \;=\; \frac{1}{g_{2,d}} \;=\; \frac{1}{g}. \tag{35}$$

Remarkably, if all parameters $g_{a,u}$ and $g_{a,d}$ are equal, then the Coulomb-like short range interactions disappear and the system becomes self-dual, i.e., $\mathcal{E}(z) = \mathcal{E}_{\mathsf{sd}}(z)$. The vanishing of the short-range interactions reminds the case of solutions at high monomer concentration and good solvents, in which the interactions act on each monomer symmetrically from any direction, so that their total effect is negligible. The situation is similar here. The term I_{C} of Eq. (27) accounts for the short range interactions and they vanish in the limit $\tau \to \infty$ and when the loops are homogeneous, see condition (35). As already mentioned, in the large τ limit the monomer distribution is not depending on the z direction implying that the short range forces due to the topological constraints acting on a monomer from above are counterbalanced by the forces acting from below. In the xy directions all legs Γ_a^u and Γ_a^d are equal under the conditions (35). It is thus likely that the short range interactions of topological origin become isotropic as in the case of polymer solutions at high monomer concentrations. Of course, what is not cancelled are the long-range interactions because they are necessary to keep the topology of the link. These long term interactions are taken into account by the self-dual contributions in Eq. (26).

4 Self-dual equations and their solutions

In this section, we derive certain solutions to the self-dual equations (32). We focus on solutions that exhibit replica symmetry breaking, specifically considering only the first replica sector. In this case Eqs. (32) take the form

$$\left[\mathsf{D}_{a,1}^{u,d} + \mathsf{i}\mathsf{D}_{a,2}^{u,d}\right] \psi_a^{u,d} = 0. \tag{36}$$

We perform in (36) the transformation

$$\psi_a^{u,d} = \sqrt{\rho_a^{u,d}}\, \mathrm{e}^{\mathsf{i}\theta_a^{u,d}}.$$

Recall that $|\psi_a^{u,d}|^2 = \rho_a^{u,d}$ offers a meaningful interpretation of the monomer densities for the four segments in two linked polymer rings. We obtain 4+1 equations after separating the real and imaginary terms:

$$\frac{1}{2}\partial_1 \log \rho_a^u - \partial_2 \theta_a^u + \sum_{b=1}^{2} C_{ab} \boldsymbol{B}_{b,2} \;=\; 0, \tag{37}$$

$$\partial_1 \theta_a^u - \sum_{b=1}^{2} C_{ab} \boldsymbol{B}_{b,1} + \frac{1}{2}\partial_2 \log \rho_a^u \;=\; 0, \tag{38}$$

$$\frac{1}{2}\partial_1 \log \rho_a^d - \partial_2 \theta_a^d - \sum_{b=1}^{2} C_{ab} \boldsymbol{B}_{b,2} \;=\; 0, \tag{39}$$

$$\partial_1 \theta_a^d + \sum_{b=1}^{2} C_{ab} \boldsymbol{B}_{b,1} + \frac{1}{2}\partial_2 \log \rho_a^d \;=\; 0, \tag{40}$$

$$\sum_{c=1}^{2} d_{ac} \epsilon^{ij} \partial_i \boldsymbol{B}_{c,j} \;=\; \sum_{b=1}^{2} C_{ba} \left(-\rho_b^u + \rho_b^d\right). \tag{41}$$

We require that

$$\theta_a^u \;=\; -\theta_a^d, \quad \rho_a^u \;=\; \frac{A_a}{\rho_a^d}, \quad \rho_1^u \;=\; \rho_2^u, \quad A_1 \;=\; A_2.$$

What remains to be solved is the following equation for ρ_1^u,

$$\Delta \log \rho_1^u \;=\; \frac{\lambda}{\pi}\left(\rho_1^u - \frac{A_1}{\rho_1^u}\right), \quad \text{where} \quad \Delta = \partial_1^2 + \partial_2^2.$$

1. Taking $A_1 > 0$ in the equation above, i.e., $A_1 = |A_1|$ and putting $\rho_1^u = \sqrt{|A_1|}\mathrm{e}^{\eta}$ one gets

$$\Delta\eta = \frac{\lambda\sqrt{|A_1|}}{\pi}\left(\mathrm{e}^{\eta} - \mathrm{e}^{-\eta}\right). \tag{42}$$

2. Taking $A_1 < 0$ in the equation above, i.e., $A_1 = -|A_1|$ and putting $\rho_1^u = \sqrt{|A_1|}\mathrm{e}^{\eta}$ one gets

$$\Delta\eta = \frac{\lambda\sqrt{|A_1|}}{\pi}\left(\mathrm{e}^{\eta} + \mathrm{e}^{-\eta}\right). \tag{43}$$

3. Taking $A_1 = 0$ and substituting $\rho_1^u = \mathrm{e}^{\phi}$ one gets $\Delta\phi = \frac{\lambda}{\pi}\mathrm{e}^{\phi}$.

Thus, the self-dual equation for ρ_1^u can be reduced to the sinh-Gordon equation when $A_1 > 0$, or to the cosh-Gordon equation when $A_1 < 0$. There is also a third possibility, $A_1 = 0$, which leads to the Liouville equation.

We search for translationally invariant solutions along $x_2 \equiv y$. This case becomes realistic in a confined geometry of a box of sides $L_1 \times L_2 \times T$ with $L_2 \gg L_1$ and $T \gg 1$.[3] Let us write Eqs. (42) and (43) in the unified form

$$\partial_1^2\eta + \partial_2^2\eta \;=\; \frac{m^2}{2}\left(\mathrm{e}^{\eta} + t\mathrm{e}^{-\eta}\right), \tag{44}$$

where $m^2 = \pi^{-1}2\lambda\sqrt{|A_1|}$, and $t = 1$ for the cosh-Gordon equation, while $t = -1$ for the sinh-Gordon equation. It can be shown in a few steps that, for translationally invariant solutions where $\eta(x_1, x_2) = \eta_1(x_1)$, Eq. (44) simplifies to the Weierstrass ordinary differential equation,

$$\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2 = 4y^3 - g_2 y - g_3. \tag{45}$$

- Indeed, in the first step one can show that

$$\frac{\mathrm{d}^2\eta_1}{\mathrm{d}x_1^2} \;=\; \frac{m^2}{2}\left(\mathrm{e}^{\eta_1} + t\mathrm{e}^{-\eta_1}\right) \quad \Leftrightarrow \quad \frac{1}{2}\left(\frac{\mathrm{d}\eta_1}{\mathrm{d}x_1}\right)^2 - \frac{m^2}{2}\left(\mathrm{e}^{\eta_1} - t\mathrm{e}^{-\eta_1}\right) \;=\; E,$$

  where E is an arbitrary constant.

- Next, substituting $v_1 = \frac{m^2}{2}\mathrm{e}^{\eta_1}$ one gets

$$\left(\frac{\mathrm{d}v_1}{\mathrm{d}x_1}\right)^2 \;=\; 2v_1^3 + 2Ev_1^2 - t\frac{m^4}{2}v_1.$$

- Finally, after one more substitution $v_1 = 2y - \frac{1}{3}E$ the above equation takes the standard Weierstrass form:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x_1}\right)^2 = 4y^3 - \left(\frac{1}{3}E^2 + t\frac{m^4}{4}\right)y + \frac{E}{3}\left(\frac{1}{9}E^2 + t\frac{m^4}{8}\right) \equiv 4y^3 - g_2 y - g_3.$$

We focus exclusively on real solutions. In this case, the solutions y to the Weierstrass equation can take on three distinct ranges, which are summarized in Table 1. Let us recall that e_1, e_2 and e_3 are the roots of the cubic polynomial $Q(y) = 4y^3 - g_2 y - g_3$, which appears in Eq. (45).

The monomer densities $\rho_a^{u,d}$, with $a = 1, 2$, for the four segments of two linked polymer rings are given by the solutions presented in Table 1 (see Table 2). It is important to emphasize that only bounded solutions are physically meaningful.

[3] The limit $T \gg 1$ corresponds to the "static" case, referring to long polymer loops extending in the z-direction.

unbounded solutions	bounded solution
$y \in [e_1, +\infty)$ for $\Delta > 0$	$y \in [e_3, e_2]$ for $\Delta > 0$
$y \in [e_2, +\infty)$ for $\Delta < 0$	

Table 1: The ranges of real solutions of the Eq. (45).

parameters	$\rho_1^u = \rho_2^u = \frac{\pi}{\lambda} v_1$	unbounded range	bounded range
$t = +1$	$\frac{2\pi}{\lambda}(y - e_2)$	$\left[\frac{2\pi}{\lambda}(e_1 - e_2), +\infty\right)$	$\left[-\frac{2\pi}{\lambda}(e_2 - e_3), 0\right]$
$t = -1$, $E > m^2$	$\frac{2\pi}{\lambda}(y - e_1)$	$[0, +\infty)$	$\left[-\frac{2\pi}{\lambda}(e_1 - e_3), -\frac{2\pi}{\lambda}(e_1 - e_2)\right]$
$t = -1$, $E < -m^2$	$\frac{2\pi}{\lambda}(y - e_3)$	$\left[\frac{2\pi}{\lambda}(e_1 - e_3), +\infty\right)$	$\left[0, \frac{2\pi}{\lambda}(e_2 - e_3)\right]$
$t = -1$, $\lvert E\rvert < m^2$	$\frac{2\pi}{\lambda}(y - e_2)$	$[0, +\infty)$	—

Table 2: The ranges of ρ_1^u and ρ_2^u.

5 Concluding remarks

The key conclusions from our research, which may be relevant to non-specialists, are as follows:

— The expectation values of observables in polymer-like systems with topological constraints can be computed using field-theoretic methods.

— The magnetic fields that arise in topologically constrained polymer systems play a key role in maintaining the topology by mediating both long- and short-range interactions. In the case of a topological state resembling a 4-plat, self-dual solutions have been derived.

— Numerical simulations are necessary to gain a deeper understanding of the self-dual case.

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References

[1] De Gennes P G 1972 Phys. Lett. **A** 38 339-340

[2] Des Cloizeaux J 1975 Journal de Physique 36 (4), 281-291

[3] Emery V J 1975 Phys. Rev. **B** 11 239

[4] Schäfer L, Witten Jr T A 1977 Jour. Chem. Phys. 66 (5) 2121

[5] Oono Y, Ohta T, Freed K F 1981 J. Chem. Phys. 74 6458

[6] Edwards S F 1968 Jour. Phys. A: General Physics (1) 15

[7] Ferrari F 2004 Phys. Lett. **A** 323 351-359

[8] Ferrari F, Paturej J, Piątek M, Zhao Y 2019 Nucl. Phys. **B** 945 114673

[9] Abbasi Taklimi N, Ferrari F, Piątek M 2024 Nucl. Phys. **B** 999 116447

[10] Dunne G V 1998 Aspects of Chern-Simons Theory (*Preprint* hep-th/9902115)

[11] Dunne G V 1994 Self-Dual Chern-Simons Theories (*Preprint* hep-th/9410065)